

Application of a \mathbb{Z}_3 -orbifold construction to the lattice vertex operator algebras associated to Niemeier lattices

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Abstract

Applying Miyamoto's \mathbb{Z}_3 -orbifold construction to the lattice vertex operator algebra associated to the Niemeier lattice with root lattice A_2^{12} and a certain automorphism of order 3, we construct a new holomorphic vertex operator algebra of central charge 24 whose Lie algebra of the weight one space is of type A_2^6 , which corresponds to No. 6 on Schellekens' list. Moreover, we classify the holomorphic VOAs of central charge 24 obtained by Miyamoto's \mathbb{Z}_3 -orbifold construction when the rank of the sublattice fixed by the automorphism of order 3 is equal to 6.

1 Introduction.

The classification of holomorphic vertex operator algebras (VOAs for short) is a fundamental problem in VOA theory. By Zhu's theory (see [Z]), their central charges are divisible by 8. We know from [DM1] that a holomorphic VOA of central charge 8 or 16 is isomorphic to a lattice VOA. Thus the next problem is to classify the holomorphic VOAs of central charge 24. In [S], Schellekens gave a list of possible 71 Lie algebra structures of the weight one spaces of holomorphic VOAs of central charge 24 (it is mysterious that the number 71 appears here; recall that 71 is the largest prime factor of the order of the Monster simple group). So the

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first step of the classification would be to construct 71 holomorphic VOAs of central charge 24, and check the Lie algebra structures of their weight one spaces.

It is well-known that the lattice VOA associated to a unimodular, positive-definite, even lattices of rank 24 (such a lattice is called a Niemeier lattice) is a holomorphic VOA of central charge 24. Because there exist exactly 24 Niemeier lattices (see, e.g., [CS, Chapter 16, Table 16.1]), we can obtain 24 holomorphic VOAs of central charge 24 in this manner. Also, we can obtain 15 holomorphic VOAs of central charge 24 by applying the \mathbb{Z}_2 -orbifold construction associated to the (-1) -isometry to lattice VOAs associated to Niemeier lattices (see [FLM, DGM]). In [La, LS1, LS2], Lam and the second named author proved that there exist exactly 56 holomorphic framed VOAs of central charge 24, including $39 (= 24 + 15)$ holomorphic VOAs mentioned above. According to Schellekens' list, there should exist $15 (= 71 - 56)$ non-framed holomorphic VOAs of central charge 24, which should correspond to:

No. in [S]	dim of the weight 1 space	Lie algebra structure
3	36	$D_{4,12}A_{2,6}$
4	36	$C_{4,10}$
6	48	$A_{2,3}^6$
8	48	$A_{5,6}C_{2,3}A_{1,2}$
9	48	$A_{4,5}^2$
11	48	$A_{6,7}$
14	60	$F_{4,6}A_{2,2}$
17	72	$A_{5,3}D_{4,3}A_{1,1}^3$
20	72	$D_{6,5}A_{1,1}^2$
21	72	$C_{5,3}G_{2,2}A_{1,1}$
27	96	$A_{8,3}A_{2,1}^2$
28	96	$E_{6,4}C_{2,1}A_{2,1}$
32	120	$E_{6,3}G_{2,1}^3$
34	120	$D_{7,3}A_{3,1}G_{2,1}$
45	168	$E_{7,3}A_{5,1}$

Here, $X_{m,n}$ denotes the simple Lie algebra of type X_m with n its level; we can easily compute the level n from the dual Coxeter number of X_m and the dimension of the weight one space (see [DM1, (3.6)]).

In [Mi], Miyamoto established a \mathbb{Z}_3 -orbifold construction to a lattice VOA and an automorphism of order 3, and then constructed a new (non-framed) holomorphic VOA of central charge 24 whose Lie algebra of the weight one space is of type $E_6G_2^3$, by applying his \mathbb{Z}_3 -orbifold construction to the lattice VOA associated to the Niemeier lattice $L = \text{Ni}(E_6^4)$ with root lattice $Q = E_6^4$ and its automorphism (of order 3) induced from a lattice automorphism σ of L of order 3; this holomorphic VOA corresponds to No. 32 on Schellekens' list.

In this paper, we also construct a new holomorphic VOA by using Miyamoto's \mathbb{Z}_3 -orbifold

construction, which corresponds to No.6 on Schellekens' list. Let us explain this result more precisely. Let Q be the root lattice of type $A_2^{12} = A_2^3 \oplus A_2^3 \oplus A_2^3 \oplus A_2^3$, and let σ be the lattice automorphism of $Q^* = (A_2^3)^* \oplus (A_2^3)^* \oplus (A_2^3)^* \oplus (A_2^3)^*$ that sends $(\mu_1, \mu_2, \mu_3, \mu_4)$ to $(x\mu_1, \mu_4, \mu_2, \mu_3)$, where x is an element of order 3 in the Weyl group of A_2^3 that acts fixed-point-freely on A_2^3 . Then, the Niemeier lattice $L = \text{Ni}(A_2^{12})$ is isomorphic to a sublattice of Q^* that is stable under σ , and induces an automorphism of the lattice VOA V_L associated to L , which we denote also by σ . Applying the \mathbb{Z}_3 -orbifold construction to V_L and $\sigma \in \text{Aut}(V_L)$, we obtain a holomorphic VOA \tilde{V}_L^σ of central charge 24, which is a \mathbb{Z}_3 -graded, simple current extension of the fixed point subVOA V_L^σ under $\sigma \in \text{Aut}(V_L)$. We prove that this holomorphic VOA corresponds to No.6 on Schellekens' list by showing that the Lie algebra of the weight one space is of type A_2^6 .

It is common to these two holomorphic VOAs obtained by the \mathbb{Z}_3 -orbifold construction that the rank of the sublattice L^σ of the Niemeier lattice L fixed by σ is equal to 6. So, in this paper, we focus on the case that the rank of the fixed point sublattice is equal to 6, and prove that if a Niemeier lattice L and its lattice automorphism σ of order 3 satisfy the condition that $\text{rank } L^\sigma = 6$, then the holomorphic VOA obtained by applying the \mathbb{Z}_3 -orbifold construction to the lattice VOA V_L associated to L and the automorphism of V_L induced from σ is isomorphic to the lattice VOA associated to a Niemeier lattice, the holomorphic VOA constructed by Miyamoto [Mi], or our holomorphic VOA mentioned above. In order to prove this theorem, we first show that if σ is contained in the Weyl group of the root lattice of L , then the holomorphic VOA \tilde{V}_L^σ is always isomorphic to a lattice VOA. Secondly, we show that if L is isomorphic to the Leech lattice, then the \mathbb{Z}_3 -orbifold construction gives the Leech lattice VOA again. Finally, we prove that if σ is not contained in the Weyl group, and L is not isomorphic to the Leech lattice, then either of the following holds: L is isomorphic to $\text{Ni}(A_2^{12})$ and σ is conjugate to the lattice automorphism mentioned above, or L is isomorphic to $\text{Ni}(E_6^4)$ and σ is conjugate to the lattice automorphism given in [Mi].

This paper is organized as follows: In Section 2, we review the construction of lattice VOAs from [LL], and that of twisted modules from [Le, DL2]. We also recall Miyamoto's \mathbb{Z}_3 -orbifold construction. In Section 3, we describe the Niemeier lattice $L = \text{Ni}(A_2^{12})$ with root lattice A_2^{12} , and introduce a lattice automorphism σ of L of order 3 based on [CS]. Then we prove that the Lie algebra of the weight one space of the holomorphic VOA \tilde{V}_L^σ obtained by the \mathbb{Z}_3 -orbifold construction is of type A_2^6 , and hence that this holomorphic VOA corresponds to No.6 on Schellekens' list. In Section 4, we classify all the holomorphic VOAs of central charge 24 obtained by applying the \mathbb{Z}_3 -orbifold construction to a Niemeier lattice VOA and the VOA automorphism induced from a lattice automorphism σ in the case that the rank of the sublattice fixed by σ is equal to 6. In Appendix, we give another proof for the result in [Mi, §5.2] by using the construction of twisted modules due to [Le, DL2].

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2 Review.

2.1 Lattice VOAs and their automorphisms. In this subsection, we review the definition of a lattice vertex operator algebra (VOA for short); for the details, see, e.g., [LL, §§6.4 and 6.5].

Let L be a positive-definite, even lattice with \mathbb{Z} -bilinear form $\langle \cdot, \cdot \rangle$. Regard $\mathfrak{h} := L \otimes_{\mathbb{Z}} \mathbb{C}$ as an abelian Lie algebra, and define its affinization to be the Lie algebra $\widehat{\mathfrak{h}} := \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ with Lie bracket given by:

$$[x \otimes t^m, y \otimes t^n] = \delta_{m+n,0} m \langle x, y \rangle \mathbf{k} \quad \text{for } x, y \in \mathfrak{h} \text{ and } m, n \in \mathbb{Z},$$

$$[\widehat{\mathfrak{h}}, \mathbf{k}] = \{0\};$$

for simplicity of notation, we denote $h \otimes t^m$ by $h(m)$ for $h \in \mathfrak{h}$ and $m \in \mathbb{Z}$. The Lie subalgebra $\widehat{\mathfrak{b}} := \mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k} \subset \widehat{\mathfrak{h}}$ acts on the one-dimensional vector space \mathbb{C} as follows: for $c \in \mathbb{C}$,

$$h(m) \cdot c = 0 \quad \text{for all } h \in \mathfrak{h} \text{ and } m \in \mathbb{Z}_{\geq 0}, \quad \mathbf{k} \cdot c = c.$$

Then we define

$$M(1) := \text{Ind}_{\widehat{\mathfrak{b}}}^{\widehat{\mathfrak{h}}} \mathbb{C}.$$

Fix a positive even integer $s \in 2\mathbb{Z}_{>0}$. Let us define $c_0 : L \times L \rightarrow \mathbb{Z}/s\mathbb{Z}$ by:

$$c_0(\alpha, \beta) = \frac{s}{2} \langle \alpha, \beta \rangle + s\mathbb{Z},$$

which is an alternating \mathbb{Z} -bilinear map. Let $\varepsilon_0 : L \times L \rightarrow \mathbb{Z}/s\mathbb{Z}$ be a 2-cocycle corresponding to $c_0 : L \times L \rightarrow \mathbb{Z}/s\mathbb{Z}$, normalized as: $\varepsilon_0(\alpha, 0) = \varepsilon_0(0, \alpha) = 0$ for all $\alpha \in L$. Let $\langle \kappa \rangle$ be the cyclic group of order s . We define a product on

$$\widehat{L} := \{(\kappa^p, e_\alpha) \mid p \in \mathbb{Z}/s\mathbb{Z}, \alpha \in L\}$$

as follows: for $p, q \in \mathbb{Z}/s\mathbb{Z}$ and $\alpha, \beta \in L$,

$$(\kappa^p, e_\alpha) \cdot (\kappa^q, e_\beta) := (\kappa^{p+q+\varepsilon_0(\alpha, \beta)}, e_{\alpha+\beta}).$$

Then, \widehat{L} is a group with (κ^0, e_0) the identity element, and is the central extension of L by the cyclic group $\langle \kappa \rangle$ of order s with $c_0 : L \times L \rightarrow \mathbb{Z}/s\mathbb{Z}$ the commutator map. The cyclic group $\langle \kappa \rangle$ acts on the one-dimensional space \mathbb{C} by: $\kappa \cdot c = \xi c$ for $c \in \mathbb{C}$, where $\xi \in \mathbb{C}$ is a primitive s -th root of unity. Define

$$\mathbb{C}\{L\} := \mathbb{C}[\widehat{L}] \otimes_{\langle \kappa \rangle} \mathbb{C},$$

where $\mathbb{C}[\widehat{L}]$ denotes the group ring of the group \widehat{L} ; remark that $\{e^\alpha := (\kappa^0, e_\alpha) \otimes 1 \mid \alpha \in L\}$ is a basis of $\mathbb{C}\{L\}$.

Now, set

$$V_L := M(1) \otimes \mathbb{C}\{L\}.$$

Then, V_L admits a VOA structure whose central charge is equal to the rank of the lattice L (which is independent of the choices of s , ε_0 , and ξ). Recall that the weight of $h_k(-n_k) \cdots h_1(-n_1)1 \otimes e^\alpha \in V_L = M(1) \otimes \mathbb{C}\{L\}$, where $h_1, \dots, h_k \in \mathfrak{h}$, $n_1, \dots, n_k \in \mathbb{Z}_{>0}$, and $\alpha \in L$, is given by:

$$n_k + \cdots + n_1 + \frac{\langle \alpha, \alpha \rangle}{2} \in \mathbb{Z}_{\geq 0}.$$

In particular, the weight one space $(V_L)_1$ of V_L is spanned by

$$\{h(-1)1 \otimes e^0 \mid h \in \mathfrak{h}\} \cup \{1 \otimes e^\alpha \mid \alpha \in \Delta\},$$

where $\Delta := \{\alpha \in L \mid \langle \alpha, \alpha \rangle = 2\}$, the set of roots in L . Denote by

$$Y(\cdot, z) : V_L \rightarrow (\text{End}_{\mathbb{C}} V_L)[[z, z^{-1}]], \quad a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

the vertex operator for V_L . For latter use, let us recall the definition of $Y(a, z)$ for some special $a \in V_L$. First, the Lie algebra $\widehat{\mathfrak{h}}$ acts on $V_L = M(1) \otimes \mathbb{C}\{L\}$ as follows: $\mathbf{k} \in \widehat{\mathfrak{h}}$ acts as the identity, and for $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$,

$$h(n)(u \otimes e^\beta) = \begin{cases} \langle h, \beta \rangle (u \otimes e^\beta) & \text{if } n = 0, \\ (h(n)u) \otimes e^\beta & \text{if } n \neq 0, \end{cases} \quad \text{for } u \in M(1) \text{ and } \beta \in L.$$

Also, for $\alpha \in L$, we define $z^\alpha \in \text{Hom}_{\mathbb{C}}(V_L, V_L[[z, z^{-1}]])$ by

$$z^\alpha(u \otimes e^\beta) := z^{\langle \alpha, \beta \rangle} (u \otimes e^\beta) \quad \text{for } u \in M(1) \text{ and } \beta \in L.$$

In addition, the group \widehat{L} acts on $V_L = M(1) \otimes \mathbb{C}\{L\}$ as follows:

$$g \cdot (u \otimes v) = u \otimes (g \cdot v) \quad \text{for } g \in \widehat{L} \text{ and } u \in M(1), v \in \mathbb{C}\{L\},$$

where the $g \cdot v$ above is given by the natural action of \widehat{L} on $\mathbb{C}\{L\}$. In particular,

$$(\kappa^0, e_\alpha) \cdot (u \otimes e^\beta) := u \otimes (\xi^{\varepsilon_0(\alpha, \beta)} e^{\alpha+\beta}) \quad \text{for } \alpha, \beta \in L \text{ and } u \in M(1).$$

We have

$$Y(h(-1)1 \otimes e^0, z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1} \quad \text{for } h \in \mathfrak{h}, \quad (2.1.1)$$

$$Y(1 \otimes e^\alpha, z) = E^-(-\alpha, z) E^+(-\alpha, z) \underbrace{(\kappa^0, e_\alpha)}_{\in \widehat{L}} z^\alpha \quad \text{for } \alpha \in L, \quad (2.1.2)$$

where

$$E^\pm(-\alpha, z) := \exp \left(\sum_{n \in \pm \mathbb{Z}_{>0}} \frac{-\alpha(n)}{n} z^{-n} \right).$$

2.2 Twisted modules over lattice VOAs. Keep the notation in §2.1. An automorphism of the lattice L is by definition, a \mathbb{Z} -module automorphism σ of L satisfying the condition that $\langle \sigma\alpha, \sigma\beta \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in L$. Denote by $\text{Aut}(L)$ the group of all lattice automorphisms of L . Let $\sigma \in \text{Aut}(L)$ be of odd order. With notation in §2.1, set $s := 2|\sigma|$, where $|\sigma|$ denotes the order σ . Replacing $\varepsilon_0 : L \times L \rightarrow \mathbb{Z}/s\mathbb{Z}$ with

$$(\alpha, \beta) \mapsto \sum_{r=0}^{|\sigma|-1} \varepsilon_0(\sigma^r \alpha, \sigma^r \beta) \quad \text{for } \alpha, \beta \in L$$

if necessary, we may assume that ε_0 is σ -invariant. We deduce that the lattice automorphism $\sigma \in \text{Aut}(L)$ naturally induces a VOA automorphism of V_L ; by abuse of notation, we denote this VOA automorphism also by $\sigma \in \text{Aut}(V_L)$. Remark that

$$\sigma(h_k(-n_k) \cdots h_1(-n_1)1 \otimes e^\alpha) = (\sigma h_k)(-n_k) \cdots (\sigma h_1)(-n_1)1 \otimes e^{\sigma\alpha}.$$

Now, we recall a construction of σ -twisted modules over the lattice VOA V_L from [DL2] and [Le]. Their results are valid for a lattice automorphism of arbitrary (finite) order, but we restrict ourselves to the case of $|\sigma| = 3$ (because it is enough for our purpose). Then, $s = 2|\sigma| = 6$. Also, set $\zeta := \xi^2 \in \mathbb{C}$, which is a primitive 3rd root of unity.

For $n \in \mathbb{Z}$, define $\mathfrak{h}_{(n)} := \{h \in \mathfrak{h} \mid \sigma h = \zeta^n h\}$; note that $\mathfrak{h}_{(n)} = \mathfrak{h}_{(n+3k)}$ for all $n, k \in \mathbb{Z}$, and $\mathfrak{h} = \mathfrak{h}_{(0)} \oplus \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)}$. Let us define the σ -twisted affine Lie algebra associated to the abelian Lie algebra \mathfrak{h} to be

$$\widehat{\mathfrak{h}}[\sigma] := \bigoplus_{n \in (1/3)\mathbb{Z}} \mathfrak{h}_{(3n)} \otimes \mathbb{C}t^n \oplus \mathbb{C}\mathbf{k}$$

with Lie bracket

$$[x \otimes t^m, y \otimes t^n] = \delta_{m+n,0} m \langle x, y \rangle \mathbf{k} \quad \text{for } m, n \in (1/3)\mathbb{Z} \text{ and } x \in \mathfrak{h}_{(3m)}, y \in \mathfrak{h}_{(3n)},$$

$$[\widehat{\mathfrak{h}}[\sigma], \mathbf{k}] = \{0\};$$

for simplicity of notation, we denote $h \otimes t^m$ by $h(m)$ for $m \in (1/3)\mathbb{Z}$ and $h \in \mathfrak{h}_{(3m)}$. The Lie subalgebra

$$\widehat{\mathfrak{b}}[\sigma] := \bigoplus_{n \in (1/3)\mathbb{Z}_{\geq 0}} \mathfrak{h}_{(3n)} \otimes \mathbb{C}t^n \oplus \mathbb{C}\mathbf{k} \subset \widehat{\mathfrak{h}}[\sigma]$$

acts on the one-dimensional vector space \mathbb{C} as follows: for $c \in \mathbb{C}$,

$$h(m) \cdot c = 0 \quad \text{for all } m \in (1/3)\mathbb{Z}_{\geq 0} \text{ and } h \in \mathfrak{h}_{(3m)}, \quad \mathbf{k} \cdot c = c.$$

Then we define

$$M(1)[\sigma] := \text{Ind}_{\widehat{\mathfrak{b}}[\sigma]}^{\widehat{\mathfrak{h}}[\sigma]} \mathbb{C}.$$

Define an alternating \mathbb{Z} -bilinear map $c_0^\sigma : L \times L \rightarrow \mathbb{Z}/6\mathbb{Z}$ by:

$$c_0^\sigma(\alpha, \beta) = \sum_{r=0}^2 (3+2r) \langle \sigma^r \alpha, \beta \rangle + 6\mathbb{Z}, \quad (2.2.1)$$

and then define $\varepsilon_0^\sigma : L \times L \rightarrow \mathbb{Z}/6\mathbb{Z}$ by (see [DL2, Remarks 2.1 and 2.2])

$$\varepsilon_0^\sigma(\alpha, \beta) := \varepsilon_0(\alpha, \beta) + \langle \sigma^{-1} \alpha, \beta \rangle \quad \text{for } \alpha, \beta \in L.$$

It can be easily checked that ε_0^σ is a σ -invariant, normalized 2-cocycle corresponding to c_0^σ . We define another product $*$ on $\widehat{L} = \{(\kappa^p, e_\alpha) \mid p \in \mathbb{Z}/6\mathbb{Z}, \alpha \in L\}$ as follows: for $p, q \in \mathbb{Z}/6\mathbb{Z}$ and $\alpha, \beta \in L$,

$$(\kappa^p, e_\alpha) * (\kappa^q, e_\beta) := (\kappa^{p+q+\varepsilon_0^\sigma(\alpha, \beta)}, e_{\alpha+\beta}).$$

Then, $(\widehat{L}, *)$ is a group with (κ_0, e_0) the identity element; we denote this group by \widehat{L}_σ . It can be easily seen that \widehat{L}_σ is the central extension of L by the cyclic group $\langle \kappa \rangle$ of order 6 with $c_0^\sigma : L \times L \rightarrow \mathbb{Z}/6\mathbb{Z}$ the commutator map.

Because ε_0^σ is σ -invariant, the lattice automorphism $\sigma \in \text{Aut}(L)$ induces a group automorphism $\sigma \in \text{Aut}(\widehat{L}_\sigma)$ defined by: $\sigma(\kappa^p, e_\alpha) = (\kappa^p, e_{\sigma\alpha})$ for $p \in \mathbb{Z}/6\mathbb{Z}$ and $\alpha \in L$.

For $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$, denote by $h_{(n)}$ the image of h under the projection $\mathfrak{h} \twoheadrightarrow \mathfrak{h}_{(n)}$; note that $h_{(0)} = (1/3) \sum_{r=0}^2 \sigma^r h$ for $h \in \mathfrak{h}$. Set

$$N := \{\alpha \in L \mid \langle \alpha, \mathfrak{h}_{(0)} \rangle = \{0\}\} = \{\alpha \in L \mid \alpha_{(0)} = 0\}; \quad (2.2.2)$$

the (second) equality follows from the fact that $\langle \sigma^r \alpha, h \rangle = \langle \alpha, h \rangle$ for all $r \in \mathbb{Z}$ and all $\alpha \in L$ and $h \in \mathfrak{h}_{(0)}$, and the fact that $\langle \cdot, \cdot \rangle$ is nondegenerate on $\mathfrak{h}_{(0)}$. Set

$$R := \{\alpha \in N \mid c_0^\sigma(\alpha, N) = 0\}, \quad M := (1 - \sigma)L; \quad (2.2.3)$$

notice that for $\alpha \in N$, the condition $c_0^\sigma(\alpha, N) = 0$ is equivalent to the condition $\langle \alpha - \sigma\alpha, N \rangle \subset 3\mathbb{Z}$ since N is stable under the action of σ . Then, $M \subset R \subset N$. Also, because

$$M \otimes_{\mathbb{Z}} \mathbb{C} = (1 - \sigma) \underbrace{(L \otimes_{\mathbb{Z}} \mathbb{C})}_{=\mathfrak{h}} = \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)} = \{h \in \mathfrak{h} \mid \langle h, \mathfrak{h}_{(0)} \rangle = \{0\}\} = N \otimes_{\mathbb{Z}} \mathbb{C}, \quad (2.2.4)$$

it follows immediately that N/M is a finite group, and hence so is N/R .

Now, for a subgroup Q of L , we set $\widehat{Q}_\sigma = \{(\kappa^p, e_\alpha) \mid p \in \mathbb{Z}/6\mathbb{Z}, \alpha \in Q\}$ (which is a subgroup of \widehat{L}_σ). Then, $\widehat{M}_\sigma \subset \widehat{R}_\sigma \subset \widehat{N}_\sigma$, and as a group,

$$\widehat{M}_\sigma = \langle (\kappa, e_0), g\sigma(g)^{-1} \mid g \in \widehat{L}_\sigma \rangle.$$

By [Le, Proposition 6.1], there exists a unique group homomorphism $\tau : \widehat{M}_\sigma \rightarrow \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ such that $\tau(\kappa, e_0) = \xi$ and $\tau(g\sigma(g)^{-1}) = \zeta^{-\sum_{r=0}^2 \langle \sigma^r \overline{g}, \overline{g} \rangle / 2}$ for $g \in \widehat{L}_\sigma$, where for $g = (\kappa^p, e_\alpha) \in$

\widehat{L}_σ , we set $\bar{g} := \alpha \in L$; observe that $\sum_{r=0}^2 \langle \sigma^r \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in L$. It is known from [Le, Proposition 6.2] (see also [DL2, Remark 4.2]) that there exists a finite-dimensional irreducible \widehat{N}_σ -module T of dimension $|N/R|^{1/2}$ on which \widehat{M}_σ acts according to $\tau : \widehat{M}_\sigma \rightarrow \mathbb{C}^\times$, i.e., $g \cdot t = \tau(g)t$ for $g \in \widehat{M}_\sigma$ and $t \in T$. Set

$$U_T := \text{Ind}_{\widehat{N}_\sigma}^{\widehat{L}_\sigma} T.$$

Remark 2.2.1 (see [Le, Section 7]). Because the image of $\tau : \widehat{M}_\sigma \rightarrow \mathbb{C}^\times$ is identical to the cyclic group $\langle \xi \rangle$ of order 6, it follows immediately that $\widehat{M}_\sigma / \ker \tau$ is a finite group. Also, since N/M is a finite group as seen above, so is $\widehat{N}_\sigma / \widehat{M}_\sigma$. Thus, $\widehat{N}_\sigma / \ker \tau$ is a finite group. Since every element in $\ker \tau$ acts on T as the identity, we conclude that each $g \in \widehat{N}_\sigma$ acts on T as a linear automorphism of finite order. In particular, the action of each $g \in \widehat{N}_\sigma$ on T is semisimple.

Now, we define

$$V_L^T := M(1)[\sigma] \otimes U_T.$$

Then we know from [DL2, Theorem 7.1] that V_L^T admits a σ -twisted V_L -module structure. Note that $V_L^T = M(1)[\sigma] \otimes U_T$ is spanned by the elements of the form: $h_k(-n_k) \cdots h_1(-n_1)1 \otimes (g \cdot t)$ with $n_1, \dots, n_k \in (1/3)\mathbb{Z}_{>0}$, $h_1 \in \mathfrak{h}_{(3n_1)}, \dots, h_k \in \mathfrak{h}_{(3n_k)}$, and $g \in \widehat{L}_\sigma$, $t \in T$. The weight of an element of the form above is given by

$$n_k + \cdots + n_1 + \rho + \frac{1}{2} \langle \bar{g}_{(0)}, \bar{g}_{(0)} \rangle \in \rho + (1/3)\mathbb{Z}_{\geq 0}, \quad (2.2.5)$$

where

$$\rho := \frac{1}{36} \sum_{r=1}^2 r(3-r) \dim \mathfrak{h}_{(r)} = \frac{1}{18} (\dim \mathfrak{h}_{(1)} + \dim \mathfrak{h}_{(2)}). \quad (2.2.6)$$

In particular,

$$(V_L^T)_\rho = \mathbb{C}1 \otimes T \quad \text{with} \quad \dim(V_L^T)_\rho = |N/R|^{1/2}; \quad (2.2.7)$$

notice that for $g \in \widehat{L}_\sigma$,

$$\langle \bar{g}_{(0)}, \bar{g}_{(0)} \rangle = 0 \iff \bar{g}_{(0)} = 0 \iff \bar{g} \in N \iff g \in \widehat{N}_\sigma.$$

Denote by

$$Y_\sigma(\cdot, z) : V_L \rightarrow (\text{End}_{\mathbb{C}} V_L^T)[[z^{1/3}, z^{-1/3}]], \quad a \mapsto Y_\sigma(a, z) = \sum_{n \in (1/3)\mathbb{Z}} a_n z^{-n-1}$$

the σ -twisted vertex operator for V_L^T . For latter use, let us recall the definition of $Y_\sigma(a, z)$ for some special $a \in V_L$. First, the Lie algebra $\widehat{\mathfrak{h}}[\sigma]$ acts on $V_L^T = M(1)[\sigma] \otimes U_T$ as follows: \mathbf{k} acts as the identity, and for $n \in (1/3)\mathbb{Z}$ and $h \in \mathfrak{h}_{(3n)}$,

$$h(n)(u \otimes (g \cdot t)) = \begin{cases} \langle h, \bar{g}_{(0)} \rangle (u \otimes (g \cdot t)) & \text{if } n = 0, \\ (h(n)u) \otimes (g \cdot t) & \text{if } n \neq 0 \end{cases} \quad (2.2.8)$$

for $u \in M(1)[\sigma]$ and $g \in \widehat{L}_\sigma$, $t \in T$. For $\alpha \in L$, define $z^{\alpha(0)} \in \text{Hom}_{\mathbb{C}}(V_L^T, V_L^T[z^{1/3}, z^{-1/3}])$ by

$$z^{\alpha(0)}(u \otimes (g \cdot t)) := z^{\langle \alpha(0), \bar{g}(0) \rangle}(u \otimes (g \cdot t)) \quad \text{for } u \in M(1)[\sigma] \text{ and } g \in \widehat{L}_\sigma, t \in T.$$

In addition, the group \widehat{L}_σ acts on V_L^T as follows:

$$g \cdot (u \otimes w) = u \otimes (g \cdot w) \quad \text{for } x \in \widehat{L}_\sigma \text{ and } u \in M(1)[\sigma], w \in U_T.$$

Now, we deduce from [DL2, (4.40) and (4.45)] that

$$Y_\sigma(h(-1)1 \otimes e^0, z) = \sum_{n \in (1/3)\mathbb{Z}} h_{(3n)}(n) z^{-n-1}; \quad (2.2.9)$$

observe that $\Delta_z(h(-1)1 \otimes e^0) = 0$, where Δ_z is defined as [DL2, (4.42)]. Also, we know from [DL2, (4.34) and (4.39)] that for $\alpha \in L$,

$$Y_\sigma(1 \otimes e^\alpha, z) = 3^{-\langle \alpha, \alpha \rangle/2} (1 - \zeta^{-1})^{\langle \sigma \alpha, \alpha \rangle} \times \\ E_\sigma^-(-\alpha, z) E_\sigma^+(-\alpha, z) \underbrace{(\kappa^0, e_\alpha)}_{\in \widehat{L}_\sigma} z^{\alpha(0) + \{\langle \alpha(0), \alpha(0) \rangle - \langle \alpha, \alpha \rangle\}/2}, \quad (2.2.10)$$

where

$$E_\sigma^\pm(-\alpha, z) := \exp \left(\sum_{n \in \pm(1/3)\mathbb{Z}_{>0}} \frac{-\alpha_{(3n)}(n)}{n} z^{-n} \right).$$

Recall that for every $a \in (V_L)_1$, the 0-th operator $a_0 \in \text{End}_{\mathbb{C}}(V_L^T)$ (i.e., the coefficient of z^{-1} in $Y_\sigma(a, z)$) is weight-preserving. In particular, the top weight space $(V_L^T)_\rho$ is stable under the action of a_0 .

Lemma 2.2.2. (1) For every $h \in \mathfrak{h}$, the 0-th operator $(h(-1)1 \otimes e^0)_0 \in \text{End}_{\mathbb{C}}(V_L^T)$ of $h(-1)1 \otimes e^0 \in (V_L)_1$ acts on the top weight space $(V_L^T)_\rho$ trivially.

(2) If $\alpha \in \Delta \setminus N$, then the 0-th operator $(1 \otimes e^\alpha)_0 \in \text{End}_{\mathbb{C}}(V_L^T)$ of $1 \otimes e^\alpha \in (V_L)_1$ acts on the top weight space $(V_L^T)_\rho$ trivially.

(3) If $\alpha \in \Delta \cap N$, then the action of the 0-th operator $(1 \otimes e^\alpha)_0 \in \text{End}_{\mathbb{C}}(V_L^T)$ of $1 \otimes e^\alpha \in (V_L)_1$ on $(V_L^T)_\rho$ is semisimple.

Proof. (1) By (2.2.9), we have $(h(-1)1 \otimes e^0)_0 = h_{(0)}(0)$. Because $(V_L^T)_\rho = \mathbb{C}1 \otimes T$ by (2.2.7), it follows immediately that $(h(-1)1 \otimes e^0)_0 = h_{(0)}(0)$ acts on the top weight space $(V_L^T)_\rho$ trivially.

(2), (3) Let $\alpha \in \Delta$, and let $1 \otimes t \in (V_L^T)_\rho = \mathbb{C}1 \otimes T$. Set $v := (\kappa^0, e_\alpha) \cdot t \in U_T$, and $d := \langle \alpha(0), \alpha(0) \rangle/2 \in (1/3)\mathbb{Z}$; remark that $d = 0$ if and only if $\alpha \in N$. By (2.2.10), $(1 \otimes e^\alpha)_0(1 \otimes t)$ is a scalar multiple of the coefficient of z^{-1} in

$$E_\sigma^-(-\alpha, z) E_\sigma^+(-\alpha, z) (\kappa^0, e_\alpha) z^{\alpha(0) + \{\langle \alpha(0), \alpha(0) \rangle - \langle \alpha, \alpha \rangle\}/2} (1 \otimes t)$$

$$\begin{aligned}
&= E_{\sigma}^{-}(-\alpha, z)E_{\sigma}^{+}(-\alpha, z)(1 \otimes v)z^{-1+d} \\
&= E_{\sigma}^{-}(-\alpha, z)(1 \otimes v)z^{-1+d} \\
&\quad \text{since } \alpha_{(3n)}(n)1 = 0 \text{ for all } n \in (1/3)\mathbb{Z}_{>0} \\
&= (1 \otimes v)z^{-1+d} + (\text{higher terms}).
\end{aligned}$$

If $\alpha \notin N$, then the coefficient of z^{-1} in $Y_{\sigma}(1 \otimes e^{\alpha}, z)(1 \otimes t)$ is equal to 0 (since $d > 0$), which implies that $(1 \otimes e^{\alpha})_0(1 \otimes t) = 0$. Thus we have proved part (2). Assume that $\alpha \in N$. Then, $(\kappa^0, e_{\alpha}) \in \widehat{N}_{\sigma}$, and $(1 \otimes e^{\alpha})_0$ sends $1 \otimes t \in (V_L^T)_{\rho} = \mathbb{C}1 \otimes T$ to a scalar multiple of $1 \otimes v = 1 \otimes ((\kappa^0, e_{\alpha}) \cdot t)$. Therefore, by Remark 2.2.1, $(1 \otimes e^{\alpha})_0$ is semisimple on $(V_L^T)_{\rho}$. Thus we have proved part (3). \square

The following lemma may be known, but we give a proof for completion.

Lemma 2.2.3. *The σ -twisted V_L -module $V_L^T = M(1)[\sigma] \otimes U_T$ is irreducible.*

Proof. Let us show that if $W \subset V_L^T$ is a nonzero σ -twisted V_L -submodule, then $W = V_L^T$.

First, we show that W contains a nonzero element of the form: $u \otimes (g \cdot t)$ for some $u \in M(1)[\sigma]$ and $g \in \widehat{L}_{\sigma}$, $t \in T$. Take a complete set $\{g_i \mid i \in I\} \subset \widehat{L}_{\sigma}$ of representatives for cosets in $\widehat{L}_{\sigma}/\widehat{N}_{\sigma}$. Then,

$$U_T = \bigoplus_{i \in I} g_i \cdot T.$$

Let $w \in W$, $w \neq 0$. There exists a finite subset J of I such that $w = \sum_{i \in J} u_i \otimes (g_i \cdot t_i)$ with some $u_i \in M(1)[\sigma]$ and $t_i \in T$ for $i \in J$. For each $h \in \mathfrak{h}_{(0)}$, we have

$$h(0)^p w = \sum_{i \in J} \langle h, \overline{g_{i(0)}} \rangle^p (u_i \otimes (g_i \cdot t_i)) \quad \text{for } 0 \leq p \leq |J| - 1. \quad (2.2.11)$$

Because $\overline{g_{i(0)}}$, $i \in I$, are all distinct (notice that $\overline{g_{i(0)}} = \overline{g_{j(0)}} \iff \overline{g_i} - \overline{g_j} \in N \iff g_i g_j^{-1} \in \widehat{N}_{\sigma}$), we can take $h \in \mathfrak{h}_{(0)}$ in such a way that $\langle h, \overline{g_{i(0)}} \rangle$, $i \in J$, are all distinct. Then the coefficient matrix of equation system (2.2.11) (which is a Vandermonde type matrix) is invertible. Therefore, for each $i \in J$, $u_i \otimes (g_i \cdot t_i)$ can be written as a linear combination of $h(0)^p w$ for $0 \leq p \leq |J| - 1$. Since $h(0) = (h(-1)1 \otimes e^0)_0$ by (2.2.9), and since W is a σ -twisted V_L -submodule by assumption, we get $u_i \otimes (g_i \cdot t_i) \in W$ for every $i \in J$.

Next, let us show that W includes the top weight space $(V_L^T)_{\rho} = \mathbb{C}1 \otimes U_T$. Take $g \in \widehat{L}_{\sigma}$ and $t \in T$ such that

$$W \cap (M(1)[\sigma] \otimes (g \cdot t)) \neq \{0\}.$$

By virtue of (2.2.8) and (2.2.9), both of W and $M(1)[\sigma] \otimes (g \cdot t)$ are $\widehat{\mathfrak{h}}[\sigma]$ -modules. Furthermore, we deduce by standard argument (as for the Fock space over the Heisenberg algebra) that $M(1)[\sigma] \otimes (g \cdot t)$ is an irreducible $\widehat{\mathfrak{h}}[\sigma]$ -module. Thus, $W \cap (M(1)[\sigma] \otimes (g \cdot t)) = M(1)[\sigma] \otimes (g \cdot t)$, which implies that

$$W \supset M(1)[\sigma] \otimes (g \cdot t).$$

In particular, $W \supset \mathbb{C}1 \otimes (g \cdot t)$. Now, for $g' = (\kappa^p, e_\alpha) \in \widehat{L}_\sigma$, we deduce, as in the proof of part (2), (3) of Lemma 2.2.2, that

$$Y(1 \otimes e^\alpha, z) \underbrace{(1 \otimes (g \cdot t))}_{\in W} = C(1 \otimes (g'g \cdot t))z^d + (\text{higher terms})$$

for some nonzero $C \in \mathbb{C}^\times$ and $d \in (1/3)\mathbb{Z}$; recall that $(\kappa, e_0) \in \widehat{L}_\sigma$ acts on U_T as a scalar multiple by ξ . Hence, $1 \otimes (g'g \cdot t) \in W$ for every $g' \in \widehat{L}_\sigma$, which implies that $1 \otimes U_T \subset W$.

Finally, since W is an $\widehat{\mathfrak{h}}[\sigma]$ -module as mentioned above, it follows immediately that $M(1)[\sigma] \otimes U_T \subset W$, and hence $W = M(1)[\sigma] \otimes U_T$. We have thus proved the lemma. \square

2.3 Miyamoto's \mathbb{Z}_3 -orbifold construction. Keep the notation in the previous subsections. Assume, in addition, that

- (i) L is unimodular, and
- (ii) $\rho = \frac{1}{18}(\dim \mathfrak{h}_{(1)} + \dim \mathfrak{h}_{(2)}) \in (1/3)\mathbb{Z}$ (see [Mi, §5]).

Assumption (i) implies that the lattice VOA V_L is holomorphic. Also we know from [DLM2, Theorem 10.3] that for each $r = 1, 2$, there exists a unique irreducible σ^r -twisted V_L -module, which we denote by $V_L(\sigma^r)$; by Lemma 2.2.3, these twisted V_L -modules are constructed by the method (due to Dong and Lepowsky) mentioned in §2.2. Recall from (2.2.5) that $V_L(\sigma)$ and $V_L(\sigma^2)$ decompose as follows:

$$V_L(\sigma) = \bigoplus_{n \in (1/3)\mathbb{Z}_{\geq 0}} V_L(\sigma)_{\rho+n}, \quad V_L(\sigma^2) = \bigoplus_{n \in (1/3)\mathbb{Z}_{\geq 0}} V_L(\sigma^2)_{\rho+n}.$$

Remark 2.3.1. We see from [DLM1, Lemma 3.7] that the restricted dual

$$V_L(\sigma)' := \bigoplus_{n \in (1/3)\mathbb{Z}_{\geq 0}} V_L(\sigma)_{\rho+n}^*$$

of $V_L(\sigma)$ admits an irreducible σ^2 -twisted V_L -module structure. By uniqueness, $V_L(\sigma^2)$ is isomorphic to $V_L(\sigma)'$.

Recall that $\rho \in (1/3)\mathbb{Z}_{\geq 0}$ by assumption (ii). Set

$$V_L(\sigma)_{\mathbb{Z}} = \bigoplus_{n \in \mathbb{Z}} V_L(\sigma)_n, \quad V_L(\sigma^2)_{\mathbb{Z}} = \bigoplus_{n \in \mathbb{Z}} V_L(\sigma^2)_n,$$

and then

$$\widetilde{V}_L^\sigma := V_L^\sigma \oplus V_L(\sigma)_{\mathbb{Z}} \oplus V_L(\sigma^2)_{\mathbb{Z}},$$

where V_L^σ is the fixed point subVOA of V_L under $\sigma \in \text{Aut}(V_L)$.

Theorem 2.3.2 ([Mi, §5]). *We can give \widetilde{V}_L^σ a VOA structure whose central charge is equal to the rank of the lattice L . Furthermore, the VOA \widetilde{V}_L^σ is C_2 -cofinite and holomorphic.*

Remark 2.3.3. In fact, the VOA \tilde{V}_L^σ is a $(\mathbb{Z}/3\mathbb{Z})$ -graded simple current extension of V_L^σ ; for the definition and properties of simple current extensions, see, e.g., [LY, §2].

Denote by

$$\tilde{Y}(\cdot, z) : \tilde{V}_L^\sigma \rightarrow (\text{End}_{\mathbb{C}} \tilde{V}_L^\sigma)[[z, z^{-1}]], \quad a \mapsto \tilde{Y}(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

the vertex operator for the VOA \tilde{V}_L^σ . Then, for $a \in V_L^\sigma$,

$$\tilde{Y}(a, z) = \begin{cases} Y(a, z) & \text{on } V_L^\sigma, \\ Y_\sigma(a, z) & \text{on } V_L(\sigma)_{\mathbb{Z}}, \\ Y_{\sigma^2}(a, z) & \text{on } V_L(\sigma^2)_{\mathbb{Z}}. \end{cases} \quad (2.3.1)$$

2.4 Proof of $R = M$ when L is unimodular. Keep the notation and setting in §2.2; recall from (2.2.2) and (2.2.3) the definitions of N , R , and M . The following proposition will be used in the proof of Proposition 3.3.3 below.

Proposition 2.4.1. *If the lattice L is unimodular, then $R = M$.*

This proposition follows from [Le, Proposition 6.2] and the uniqueness of σ -twisted V_L -modules (see §2.3), but we give a lattice theoretical proof for it here.

Proof of Proposition 2.4.1. First, let us show the following claims.

Claim 1 (see also [Ma, Proposition 1.1.3]). *The quotient \mathbb{Z} -module L/N is free.*

Proof of Claim 1. Assume that $\alpha + N \in L/N$, $\alpha \in L$, is finite order, i.e., $p\alpha + N = N$ for some $p \in \mathbb{Z}_{\geq 1}$. Then, $p\alpha \in N$, and hence $\alpha \in N \otimes_{\mathbb{Z}} \mathbb{C}$. Since $N \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)}$ by (2.2.4), we deduce that $L \cap (N \otimes_{\mathbb{Z}} \mathbb{C}) = N$. Therefore we get $\alpha \in N$, which implies that $\alpha + N = N$. This proves Claim 1. ■

Let $P : \mathfrak{h} \rightarrow W$ be the projection from $\mathfrak{h} = \mathfrak{h}_{(0)} \oplus \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)}$ onto $N \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)}$.

Claim 2 (see also [Ma, Proposition 1.3.4]). *We have*

$$P(L) = N^* := \{w \in N \otimes_{\mathbb{Z}} \mathbb{C} \mid \langle w, N \rangle \subset \mathbb{Z}\}.$$

Proof of Claim 2. It follows immediately from the definition of N that for every $\alpha \in L$,

$$\langle P(\alpha), N \rangle = \langle \alpha_{(1)} + \alpha_{(2)}, N \rangle = \langle \alpha_{(0)} + \alpha_{(1)} + \alpha_{(2)}, N \rangle = \langle \alpha, N \rangle \subset \mathbb{Z}.$$

Thus we get $P(L) \subset N^*$. Let us show the reverse inclusion. By Claim 1, there exists a \mathbb{Z} -basis $\{\mathbf{e}_i \mid 1 \leq i \leq \text{rank } L\}$ such that $\{\mathbf{e}_i \mid 1 \leq i \leq \text{rank } N\}$ is a \mathbb{Z} -basis of N . Let $\{\mathbf{e}_i^* \mid 1 \leq i \leq \text{rank } L\} \subset L^*$ be the dual basis. Because L is unimodular, $\mathbf{e}_i^* \in L$ for all

$1 \leq i \leq \text{rank } L$. Furthermore we deduce that $\{P(\mathbf{e}_i^*) \mid 1 \leq i \leq \text{rank } N\}$ forms a basis of N^* . Therefore we obtain $P(L) \supset N^*$, and hence $P(L) = N^*$, as desired. \blacksquare

Remark that $1 - \sigma$ gives a linear automorphism of $N \otimes_{\mathbb{Z}} \mathbb{C}$, since $N \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)}$ as seen above, and since N is stable under the action of σ .

Claim 3. *It holds that*

- (1) $(1 - \sigma^2)w = 3(1 - \sigma)^{-1}w$ and $(1 - \sigma)^2w = -3\sigma w$ for all $w \in N \otimes_{\mathbb{Z}} \mathbb{C}$;
- (2) $(1 - \sigma^2)N^* = (1 - \sigma)N^*$;
- (3) $|N/(1 - \sigma)N| = 3^{(\text{rank } N)/2}$.

Proof of Claim 3. Part (1) follows immediately from the fact that $1 + \sigma + \sigma^2 = 0$ on $N \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)}$. Because N^* is stable under the action of σ , we get

$$(1 - \sigma^2)N^* = -(1 - \sigma)\sigma^2N^* = -(1 - \sigma)N^* = (1 - \sigma)N^*,$$

thereby completing a proof of part (2). Let us show part (3). By (1), we have $(1 - \sigma)^2N = -3\sigma N = 3N$, and hence $|N/(1 - \sigma)^2N| = |N/3N| = 3^{\text{rank } N}$. Since $1 - \sigma$ induces an isomorphism from $N/(1 - \sigma)N$ to $(1 - \sigma)N/(1 - \sigma)^2N$, we obtain

$$3^{\text{rank } N} = |N/(1 - \sigma)^2N| = |N/(1 - \sigma)N| \cdot |(1 - \sigma)N/(1 - \sigma)^2N| = |N/(1 - \sigma)N|^2,$$

and hence $|N/(1 - \sigma)N| = 3^{(\text{rank } N)/2}$, as desired. Thus we have proved Claim 3. \blacksquare

Now, by (2.2.3) and the comment after it, we have $R = \{\alpha \in N \mid \langle \alpha - \sigma\alpha, N \rangle \subset 3\mathbb{Z}\}$. Thus,

$$\begin{aligned} R &= 3(1 - \sigma)^{-1}N^* \cap N = (1 - \sigma^2)N^* \cap N \quad \text{by Claim 3 (1)} \\ &= (1 - \sigma)N^* \cap N \quad \text{by Claim 3 (2)} \\ &= (1 - \sigma)P(L) \cap N \quad \text{by Claim 2.} \end{aligned}$$

Since $1 - \sigma = 0$ on $\mathfrak{h}_{(0)}$, it is obvious that $(1 - \sigma)L = (1 - \sigma)P(L)$. Therefore, $R = (1 - \sigma)P(L) \cap N = (1 - \sigma)L \cap N = M \cap N = M$. Thus we have proved the proposition. \square

3 Holomorphic VOA corresponding to No.6 on Schellekens' list.

3.1 Niemeier lattice with root lattice A_2^{12} and its automorphism of order 3. A Niemeier lattice is by definition, a unimodular, positive-definite, even lattice of rank 24; for the list of all Niemeier lattices, see [CS, Chapter 16, Table 16.1]. In this subsection, we recall the definition of the Niemeier lattice $\text{Ni}(A_2^{12})$ with A_2^{12} the root lattice, and define a lattice automorphism $\sigma \in \text{Aut}(\text{Ni}(A_2^{12}))$ of order 3.

Following [CS, Chapter 4, §6.1], we set

$$A_2 := \{(x_0, x_1, x_2) \in \mathbb{Z}^3 \mid x_0 + x_1 + x_2 = 0\},$$

$$[0] = (0, 0, 0) \in A_2^*, \quad [1] = \frac{1}{3}(1, 1, -2) \in A_2^*, \quad [2] = \frac{1}{3}(2, -1, -1) \in A_2^*.$$

If we set

$$\overline{[0]} = [0] + A_2, \quad \overline{[1]} = [1] + A_2, \quad \overline{[2]} = [2] + A_2,$$

then $A_2^*/A_2 = \{\overline{[\ell]} \mid \ell = 0, 1, 2\}$, and the additive group A_2^*/A_2 is naturally isomorphic to $\mathbb{F}_3 := \mathbb{Z}/3\mathbb{Z}$ (see the comment after [CS, Chapter 4, §6.1, (55)]). Define Q to be the direct sum A_2^{12} of 12 copies of A_2 ; following [CS, Chapter 10, §1.5], we use $\Omega := \{\infty, 0, 1, \dots, 10\}$ for the index set of the coordinate for Q , that is, $Q = \{(\alpha_i)_{i \in \Omega} \mid \alpha_i \in A_2 \text{ for } i \in \Omega\}$. Since $Q^* = (A_2^*)^{12}$, it follows that

$$Q^*/Q = \{(\overline{[\ell_i]})_{i \in \Omega} \mid \ell_i = 0, 1, 2 \text{ for } i \in \Omega\},$$

which is isomorphic, as an additive group, to \mathbb{F}_3^{12} . For $\overline{[\ell]} \in A_2^*/A_2$ and $i \in \Omega$, define $\overline{[\ell]}^{(i)}$ to be the element $(\overline{[\ell_i]})_{i \in \Omega} \in Q^*/Q$ with $\ell_i = \ell$ and $\ell_j = 0$ for all $j \in \Omega, j \neq i$. If we regard Q^*/Q as a 12-dimensional vector space over \mathbb{F}_3 , then $\{\overline{[1]}^{(i)} \mid i \in \Omega\}$ forms a basis of Q^*/Q , which we identify with the standard orthonormal basis of \mathbb{F}_3^{12} . Denote by $\mathfrak{S}(\Omega)$ the symmetric group of Ω ; each element $g \in \mathfrak{S}(\Omega)$ acts linearly on the vector space Q/Q^* by: $g \cdot \overline{[1]}^{(i)} = \overline{[1]}^{(g(i))}$ for $i \in \Omega$. Define $\nu \in \mathfrak{S}(\Omega)$ by: $\nu = (\infty)(X9876543210)$, where X denotes 10, that is,

$$\nu(i) = \begin{cases} 10 & \text{if } i = 0, \\ i - 1 & \text{if } 1 \leq i \leq 10, \\ \infty & \text{if } i = \infty. \end{cases}$$

Set $\Theta := \{0, 1, 3, 4, 5, 9\} \subset \Omega$, and define

$$w_0 := \sum_{i \in \Omega \setminus \Theta} \overline{[1]}^{(i)} - \sum_{j \in \Theta} \overline{[1]}^{(j)} = \sum_{i \in \Omega \setminus \Theta} \overline{[1]}^{(i)} + 2 \sum_{j \in \Theta} \overline{[1]}^{(j)},$$

$$w_i := \nu^i \cdot w_0 \quad \text{for } 0 \leq i \leq 10, \quad w_\infty := \sum_{i \in \Omega} \overline{[1]}^{(i)}.$$

Theorem 3.1.1 ([CS, Chapter 10, Theorems 2 and 3]). (1) Define \mathcal{C}_{12} to be the subspace of Q^*/Q spanned by $\{w_i \mid i \in \Omega\}$. Then, \mathcal{C}_{12} is isomorphic to the ternary Golay code in \mathbb{F}_3^{12} .

(2) The subspace \mathcal{C}_{12} is 6-dimensional with $\{w_\infty, w_1, w_3, w_4, w_5, w_9\}$ a basis.

(3) The subspace \mathcal{C}_{12} is stable under the action of $\nu \in \mathfrak{S}(\Omega)$.

(4) Define $\delta \in \mathfrak{S}(\Omega)$ by: $\delta = (\infty)(0)(1)(2X)(34)(59)(67)(8)$. Then, \mathcal{C}_{12} is stable under the action of $\delta \in \mathfrak{S}(\Omega)$.

(5) Set $\sigma_0 := \nu^{-1} \circ \delta$. Then, $\sigma_0 = (\infty)(4)(7)(012)(35X)(689)$. In particular, the order of σ_0 is equal to 3.

We now set (see [CS, Chapter 18, §4, II])

$$(Q \subset) \quad \text{Ni}(A_2^{12}) := \bigsqcup_{C \in \mathcal{C}_{12}} C \quad (\subset Q^*).$$

From now on, we arrange the coordinate of $Q^* = (A_2^*)^{12}$ as follows:

$$(\mu_i)_{i \in \Omega} = \left(\underbrace{\mu_\infty, \mu_4, \mu_7}_{\in (A_2^*)^3} \mid \underbrace{\mu_0, \mu_3, \mu_6}_{\in (A_2^*)^3} \mid \underbrace{\mu_1, \mu_5, \mu_8}_{\in (A_2^*)^3} \mid \underbrace{\mu_2, \mu_{10}, \mu_9}_{\in (A_2^*)^3} \right).$$

We first define an automorphism σ_0 of Q^* of order 3 by:

$$(\mu_{\infty 47} \mid \mu_{036} \mid \mu_{158} \mid \mu_{2X9}) \xrightarrow{\sigma_0} (\mu_{\infty 47} \mid \mu_{2X9} \mid \mu_{036} \mid \mu_{158})$$

for $\mu_{\infty 47}, \mu_{036}, \mu_{158}, \mu_{2X9} \in (A_2^*)^3$. Then we deduce from Theorem 3.1.1 (5) that σ_0 stabilizes the Niemeier lattice $\text{Ni}(A_2^{12})$, and hence $\sigma_0 \in \text{Aut}(\text{Ni}(A_2^{12}))$. Next, let φ denote the automorphism of A_2^* defined by: $(x_0, x_1, x_2) \mapsto (x_2, x_0, x_1)$; note that $\varphi(\overline{[\ell]}) = \overline{[\ell]}$ for every $\ell = 0, 1, 2$. So, if we define an automorphism σ_1 of Q^* (of order 3) by:

$$(\mu_\infty, \mu_4, \mu_7 \mid \mu_{036} \mid \mu_{158} \mid \mu_{2X9}) \xrightarrow{\sigma_1} (\varphi(\mu_\infty), \varphi(\mu_4), \varphi(\mu_7) \mid \mu_{036} \mid \mu_{158} \mid \mu_{2X9})$$

for $\mu_\infty, \mu_4, \mu_7 \in A_2^*$ and $\mu_{036}, \mu_{158}, \mu_{2X9} \in (A_2^*)^3$, then σ_1 stabilizes the Niemeier lattice $\text{Ni}(A_2^{12})$, and hence $\sigma_1 \in \text{Aut}(\text{Ni}(A_2^{12}))$. Set

$$\sigma := \sigma_0 \circ \sigma_1.$$

The fixed point subspace $\mathfrak{h}_{(0)}$ of $\mathfrak{h} = \text{Ni}(A_2^{12}) \otimes \mathbb{C}$ under the σ above is identical to

$$\{(\mathbf{0} \mid \mathbf{h} \mid \mathbf{h} \mid \mathbf{h}) \mid \mathbf{h} \in A_2^3 \otimes \mathbb{C} = (A_2 \otimes \mathbb{C})^3\},$$

where $\mathbf{0}$ denotes the zero vector in $A_2^3 \otimes \mathbb{C} = (A_2 \otimes \mathbb{C})^3$. Thus, $\dim \mathfrak{h}_{(0)} = 2 \times 3 = 6$, and hence

$$\rho = \frac{1}{18}(\dim \mathfrak{h}_{(1)} + \dim \mathfrak{h}_{(2)}) = \frac{1}{18}(24 - \dim \mathfrak{h}_{(0)}) = 1. \quad (3.1.1)$$

Also, remark that

$$\mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)} = N \otimes_{\mathbb{Z}} \mathbb{C} = \{(\mathbf{h} \mid \mathbf{h}_1 \mid \mathbf{h}_2 \mid -\mathbf{h}_1 - \mathbf{h}_2) \mid \mathbf{h}, \mathbf{h}_1, \mathbf{h}_2 \in A_2^3 \otimes \mathbb{C}\}. \quad (3.1.2)$$

3.2 Main result in §3. First, let us recall the following basic fact: let $V = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n$ be a VOA with $a \in V \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in \text{End}_{\mathbb{C}}(V)[[z, z^{-1}]]$ the vertex operator. If $\dim V_0 = 1$, then the weight one space V_1 of V carries a Lie algebra structure with Lie bracket given by: $[a, b] = a_0 b$ for $a, b \in V_1$.

Now, keep the notation and setting in §3.1. By (3.1.1), we can apply Theorem 2.3.2 to $L = \text{Ni}(A_2^{12})$ and the σ above, and obtain a $(C_2$ -cofinite) holomorphic VOA \widetilde{V}_L^σ of central charge 24; note that $\dim(\widetilde{V}_L^\sigma)_0 = \dim(V_L^\sigma)_0 = 1$. We are now ready to state our main result in this paper.

Theorem 3.2.1. *Keep the notation and setting above. The Lie algebra $(\tilde{V}_L^\sigma)_1$ is isomorphic to the semisimple Lie algebra of type A_2^6 , and the levels of simple components of $(\tilde{V}_L^\sigma)_1$ are all equal to 3. Therefore, \tilde{V}_L^σ corresponds to No. 6 on Schellekens' list [S, Table 1].*

For the definition of the “level” of a simple component, see, e.g., [DM1, (3.2) and (3.3)]. We can easily compute it by using the formula [DM1, (3.6)]; in our case, the dimension of $(\tilde{V}_L^\sigma)_1$ is equal to $8 \times 6 = 48$, and the dual Coxeter numbers of simple components (i.e., A_2 's) are all equal to 3. Hence the levels of simple components are all equal to $3 \times 24 / (48 - 24) = 3$.

3.3 Proof of Theorem 3.2.1. Recall that

$$(\tilde{V}_L^\sigma)_1 = (V_L^\sigma)_1 \oplus V_L(\sigma)_1 \oplus V_L(\sigma^2)_1.$$

We should remark that this decomposition makes $(\tilde{V}_L^\sigma)_1$ a $(\mathbb{Z}/3\mathbb{Z})$ -graded Lie algebra; in particular, $(V_L^\sigma)_1$ is a Lie subalgebra of $(\tilde{V}_L^\sigma)_1$, and each of $V_L(\sigma)_1$ and $V_L(\sigma^2)_1$ is a $(V_L^\sigma)_1$ -module via the adjoint action.

First, let us determine the Lie algebra structure of $(V_L^\sigma)_1$. For $h \in A_2 \otimes \mathbb{C}$, define $h^{(i)}$ to be $(h_i)_{i \in \Omega} \in \mathfrak{h} = L \otimes \mathbb{C}$ with $h_i = h$ and $h_j = 0$ for all $j \in \Omega$, $j \neq i$, and set

$$\begin{aligned} h^{(012)} &:= h^{(0)} + h^{(1)} + h^{(2)} = (0, 0, 0 \mid h, 0, 0 \mid h, 0, 0 \mid h, 0, 0), \\ h^{(35X)} &:= h^{(3)} + h^{(5)} + h^{(10)} = (0, 0, 0 \mid 0, h, 0 \mid 0, h, 0 \mid 0, h, 0), \\ h^{(689)} &:= h^{(6)} + h^{(8)} + h^{(9)} = (0, 0, 0 \mid 0, 0, h \mid 0, 0, h \mid 0, 0, h); \end{aligned}$$

note that $h^{(012)} = h^{(0)} + \sigma h^{(0)} + \sigma^2 h^{(0)}$, and so on. Observe that the set Δ of roots in L is identical to $\{\alpha^{(i)} \mid \alpha \in \Delta(A_2), i \in \Omega\}$, where $\Delta(A_2)$ is the set of roots for A_2 . We see that Δ is stable under $\sigma \in \text{Aut}(L)$, and the action of σ on Δ is fixed point free. Define $\mathfrak{g}^{(012)}$ to be the subspace spanned by $\{h^{(012)}(-1) \otimes e^0 \mid h \in A_2 \otimes \mathbb{C}\}$ and $\{1 \otimes e^{\alpha^{(0)}} + 1 \otimes e^{\alpha^{(1)}} + 1 \otimes e^{\alpha^{(2)}} \mid \alpha \in \Delta(A_2)\}$, and define $\mathfrak{g}^{(35X)}$ and $\mathfrak{g}^{(689)}$ in exactly the same way as $\mathfrak{g}^{(012)}$ with 0, 1, 2 replaced by 3, 5, 10 and 6, 8, 9, respectively. Also, define

$$\mathfrak{a} := \text{Span}_{\mathbb{C}}\{1 \otimes e^{\alpha^{(i)}} + 1 \otimes e^{\varphi(\alpha)^{(i)}} + 1 \otimes e^{\varphi^2(\alpha)^{(i)}} \mid \alpha \in \Delta(A_2), i = \infty, 4, 7\};$$

note that $\dim \mathfrak{a} = 2 \times 3 = 6$ since $|\Delta(A_2)/\varphi| = 2$.

Lemma 3.3.1. *As a vector space,*

$$(V_L^\sigma)_1 = \mathfrak{g}^{(012)} \oplus \mathfrak{g}^{(35X)} \oplus \mathfrak{g}^{(689)} \oplus \mathfrak{a}, \tag{3.3.1}$$

and hence $\dim(V_L^\sigma)_1 = 8 \times 3 + 6 = 30$. Furthermore, $\mathfrak{g}^{(012)}$, $\mathfrak{g}^{(35X)}$, and $\mathfrak{g}^{(689)}$ are ideals of the (whole) Lie algebra $(\tilde{V}_L^\sigma)_1$ isomorphic to the simple Lie algebra of type A_2 . In addition, \mathfrak{a} is an abelian Lie subalgebra, and the (adjoint) actions of an element of \mathfrak{a} on $V_L(\sigma)_1$ and $V_L(\sigma^2)_1$ are semisimple.

Proof. We can easily check (3.3.1). Let us show the assertion for $\mathfrak{g}^{(012)}$; we can show the assertions for $\mathfrak{g}^{(35X)}$ and $\mathfrak{g}^{(689)}$ similarly. It follows immediately from the definition of the vertex operator for V_L (see (2.1.1) and (2.1.2)) that $\mathfrak{g}^{(012)}$ is a Lie subalgebra isomorphic to the simple Lie algebra of type A_2 , and

$$[\mathfrak{g}^{(012)}, \mathfrak{g}^{(35X)}] = [\mathfrak{g}^{(012)}, \mathfrak{g}^{(689)}] = [\mathfrak{g}^{(012)}, \mathfrak{a}] = \{0\}.$$

Let us show that $[a, b] = a_0b = 0$ for all $a \in \mathfrak{g}^{(012)}$ and $b \in V_L(\sigma)_1$. Assume that $a = h^{(012)}(-1) \otimes e^0$ for some $h \in A_2 \otimes \mathbb{C}$. Since $V_L(\sigma)_1$ is the top weight space of $V_L(\sigma)$ by (3.1.1), it follows immediately from Lemma 2.2.2 (1), along with (2.3.1), that $a_0b = 0$ for all $b \in V_L(\sigma)_1$. Assume that $a = 1 \otimes e^{\alpha^{(0)}} + 1 \otimes e^{\alpha^{(1)}} + 1 \otimes e^{\alpha^{(2)}}$ for some $\alpha \in \Delta(A_2)$. Notice that $\alpha^{(i)}$, $i = 0, 1, 2$, is not contained in N by (3.1.2). Thus we see from Lemma 2.2.2 (2), along with (2.3.1), that $a_0b = 0$ for all $b \in V_L(\sigma)_1$. Similarly, we can show that $[a, b] = a_0b = 0$ for all $a \in \mathfrak{g}^{(012)}$ and $b \in V_L(\sigma^2)_1$. Thus we have proved that $\mathfrak{g}^{(012)}$ is an ideal of $(\widetilde{V}_L^\sigma)_1$.

Now, it can be easily checked by the definition of the vertex operator for V_L (see (2.1.2)) that \mathfrak{a} is an abelian Lie subalgebra. Let (and fix) $\alpha \in \Delta(A_2)$, and $i = \infty, 4, 7$. Note that $\varphi^r(\alpha)^{(i)} \in N$ by (3.1.2). Hence we see from Lemma 2.2.2 (3) that $(1 \otimes e^{\varphi^r(\alpha)^{(i)}})_0$ is semisimple on $V(\sigma)_1$ for each $r = 0, 1, 2$. Since $\langle \varphi^r(\alpha), \varphi^{r+1}(\alpha) \rangle = -1$ for every $r = 0, 1, 2$, it follows immediately from the definition (2.2.1) of the commutator map c_0^σ for \widehat{L}_σ that $c_0^\sigma(\varphi^r(\alpha)^{(i)}, \varphi^{r+1}(\alpha)^{(i)}) = 0$ for every $r = 0, 1, 2$. Thus, $(\kappa^0, e_{\varphi^r(\alpha)^{(i)}}) \in \widehat{L}_\sigma$, $r = 0, 1, 2$, commute with each other, and hence so are $(1 \otimes e^{\varphi^r(\alpha)^{(i)}})_0 \in \text{End}_{\mathbb{C}}(V(\sigma)_1)$, $r = 0, 1, 2$, because $(1 \otimes e^{\varphi^r(\alpha)^{(i)}})_0$ is identical to a scalar multiple of the action of $(\kappa^0, e_{\varphi^r(\alpha)^{(i)}})$ on $V(\sigma)_1$ (see the proof of Lemma 2.2.2 (3)). Therefore we conclude that $(1 \otimes e^{\alpha^{(i)}})_0 + (1 \otimes e^{\varphi(\alpha)^{(i)}})_0 + (1 \otimes e^{\varphi^2(\alpha)^{(i)}})_0$ is also semisimple on $V(\sigma)_1$. Similarly, we can show that $(1 \otimes e^{\alpha^{(i)}})_0 + (1 \otimes e^{\varphi(\alpha)^{(i)}})_0 + (1 \otimes e^{\varphi^2(\alpha)^{(i)}})_0$ is semisimple also on $V(\sigma^2)_1$. This completes the proof of the lemma. \square

Remark 3.3.2. It follows from [DM1, Theorem 3] and Lemma 3.3.1 that the Lie algebra $(\widetilde{V}_L^\sigma)_1$ is semisimple.

Define N and R, M as in (2.2.2) and (2.2.3) for $L = \text{Ni}(A_2^{12})$ and this σ . Since L is unimodular, it follows immediately from Proposition 2.4.1 that $R = M$.

Proposition 3.3.3. *It holds that $|N/R| = |N/M| = 3^4$.*

A proof of this proposition will be given in the next subsection. As an immediate consequence, we obtain the following.

Corollary 3.3.4. *We have $\dim V(\sigma)_1 = \dim V(\sigma^2)_1 = 9$, and hence $\dim(\widetilde{V}_L^\sigma)_1 = 30 + 9 \times 2 = 48$.*

Proof. By (2.2.7), along with $\rho = 1$ by (3.1.1), we have $\dim V(\sigma)_1 = (3^4)^{1/2} = 3^2$. Further, it follows from Remark 2.3.1 that $\dim V(\sigma^2)_1 = \dim V(\sigma)_1$, and hence $\dim V(\sigma^2)_1 = 3^2$. Thus we have proved the corollary. \square

Here we set

$$\mathfrak{g} := (\tilde{V}_L^\sigma)_1 / (\mathfrak{g}^{(012)} \oplus \mathfrak{g}^{(35X)} \oplus \mathfrak{g}^{(689)}),$$

and denote by $\pi : (\tilde{V}_L^\sigma)_1 \twoheadrightarrow \mathfrak{g}$ the canonical projection. Then, \mathfrak{g} is a semisimple Lie algebra satisfying the following conditions:

- (i) $\dim \mathfrak{g} = 24$;
- (ii) \mathfrak{g} contains an abelian subalgebra $\pi(\mathfrak{a})$ of dimension 6;
- (iii) \mathfrak{g} is a $(\mathbb{Z}/3\mathbb{Z})$ -graded Lie algebra: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_0 = \pi(\mathfrak{a})$, $\mathfrak{g}_1 = \pi(V(\sigma)_1)$, and $\mathfrak{g}_2 = \pi(V(\sigma^2)_1)$;
- (iv) The adjoint actions of $\mathfrak{g}_0 = \pi(\mathfrak{a})$ on \mathfrak{g}_1 and \mathfrak{g}_2 are semisimple.

Proposition 3.3.5. *The Lie algebra \mathfrak{g} above is isomorphic to the semisimple Lie algebra of type A_2^3 .*

Proof. We see that \mathfrak{g}_0 is a toral subalgebra in the sense of [H, §8.1]. So, by [H, §15.3, Corollary], there exists a Cartan subalgebra \mathfrak{H} of \mathfrak{g} containing \mathfrak{g}_0 . Thus the rank of \mathfrak{g} is greater than or equal to $6 = \dim \mathfrak{g}_0$. Combining this fact and (i), we deduce that \mathfrak{g} is isomorphic to the semisimple Lie algebra of type

$$A_1^8, \quad A_2^3, \quad \text{or} \quad B_2 \oplus A_2 \oplus A_1^2. \quad (3.3.2)$$

By (iv), \mathfrak{g}_1 and \mathfrak{g}_2 admit the weight space decompositions with respect to \mathfrak{g}_0 :

$$\mathfrak{g}_1 = \bigoplus_{\alpha \in \mathfrak{g}_0^*} \mathfrak{g}_1^\alpha, \quad \mathfrak{g}_2 = \bigoplus_{\alpha \in \mathfrak{g}_0^*} \mathfrak{g}_2^\alpha,$$

where $\mathfrak{g}_j^\alpha = \{x \in \mathfrak{g}_j \mid [h, x] = \alpha(h)x \text{ for } h \in \mathfrak{g}_0\}$ for $j = 1, 2$ and $\alpha \in \mathfrak{g}_0^*$.

Claim 1. The Lie algebra \mathfrak{g} is not isomorphic to the semisimple Lie algebra of type A_1^8 .

Suppose that the assertion is false. Let \mathfrak{H} be the Cartan subalgebra of $\mathfrak{g} \cong A_1^8$ containing \mathfrak{g}_0 . Because \mathfrak{H} is abelian, it follows that $\mathfrak{H} \subset \mathfrak{g}_0 \oplus \mathfrak{g}_1^0 \oplus \mathfrak{g}_2^0$. Since $\mathfrak{g}_0 \subsetneq \mathfrak{H}$, there exists $x \in \mathfrak{H}$, $x \neq 0$, such that $x \in \mathfrak{g}_1^0 \oplus \mathfrak{g}_2^0$; write it as $x = x_1 + x_2$ with $x_j \in \mathfrak{g}_j^0$ for $j = 1, 2$. We can take $y \in \mathfrak{g} \setminus \{0\}$, such that for some $k \in \mathbb{C}^\times$ and $\gamma \in \mathfrak{g}_0^*$,

$$\begin{cases} [x, y] = ky, \\ [h, y] = \gamma(h)y \quad \text{for all } h \in \mathfrak{g}_0. \end{cases}$$

Write y as: $y = h' + y_1 + y_2$ with $h' \in \mathfrak{g}_0$ and $y_j \in \mathfrak{g}_j$ for $j = 1, 2$.

Assume first that $\gamma \neq 0$. If we take $h \in \mathfrak{g}_0$ such that $\gamma(h) \neq 0$, then

$$\gamma(h)y = [h, y] = [h, y_1] + [h, y_2].$$

Since $\gamma(h) \neq 0$, it follows immediately that $h' = 0$. Also we see that $y_j \in \mathfrak{g}_j^\gamma$ for $j = 1, 2$. Note that $[x_1, y_2] \in \mathfrak{g}_0$ by (iii). Hence we have

$$0 = [h, [x_1, y_2]] = \underbrace{(0 + \gamma)(h)}_{\neq 0} [x_1, y_2],$$

which implies that $[x_1, y_2] = 0$. Similarly, $[x_2, y_1] = 0$. Combining these, we obtain

$$ky = [x, y] = [x_2, y_2] + [x_1, y_1],$$

and hence $ky_1 = [x_2, y_2]$, $ky_2 = [x_1, y_1]$ by (iii). Substituting the first equality into the second equality, we obtain

$$ky_2 = \frac{1}{k}[x_1, [x_2, y_2]] = \frac{1}{k}[\underbrace{[x_1, x_2]}_{\in \mathfrak{g}_0}, y_2] + \frac{1}{k}[x_2, \underbrace{[x_1, y_2]}_{=0}] = \frac{1}{k}\gamma([x_1, x_2])y_2.$$

Similarly, substituting the second equality into the first equality, we obtain

$$ky_1 = \frac{1}{k}[x_2, [x_1, y_1]] = \frac{1}{k}[\underbrace{[x_2, x_1]}_{\in \mathfrak{g}_0}, y_1] + \frac{1}{k}[x_1, \underbrace{[x_2, y_1]}_{=0}] = \frac{1}{k}\gamma([x_2, x_1])y_1.$$

If $y_1 \neq 0$ and $y_2 \neq 0$, then $\gamma([x_1, x_2]) = k^2 = \gamma([x_2, x_1]) = -\gamma([x_1, x_2])$, which is a contradiction. Thus, $y_1 = 0$ or $y_2 = 0$. If $y_1 = 0$ (resp., $y_2 = 0$), then $ky_2 = [x_1, y_1] = 0$ (resp., $ky_1 = [x_2, y_2] = 0$), and hence $y = 0$, which contradicts the assumption that $y \neq 0$.

Assume next that $\gamma = 0$; as above, we see that $y_j \in \mathfrak{g}_j^0$ for $j = 1, 2$. Since $x_j \in \mathfrak{g}_j^0$ for $j = 1, 2$, it follows that

$$ky = [x, y] = [x_1, y_2] + [x_2, y_1] + [x_2, y_2] + [x_1, y_1], \quad (3.3.3)$$

and hence $ky_1 = [x_2, y_2]$, $ky_2 = [x_1, y_1]$ by (iii). Substituting the first equality into the second equality, we obtain

$$ky_2 = \frac{1}{k}[x_1, [x_2, y_2]] = \frac{1}{k}[\underbrace{[x_1, x_2]}_{\in \mathfrak{g}_0}, y_2] + \frac{1}{k}[x_2, \underbrace{[x_1, y_2]}_{\in \mathfrak{g}_0}] = 0,$$

since $y_2, x_2 \in \mathfrak{g}_2^0$. Thus, $y_2 = 0$. Similarly, it can be easily seen that $y_1 = 0$. Hence, $y = h' \in \mathfrak{g}_0$. However, by (3.3.3) and the assumption that $x \in \mathfrak{H}$, we get $ky = [x, y] = 0$, which contradicts the assumption that $k \neq 0$ and $y \neq 0$. This proves Claim 1.

Claim 2. The Lie algebra \mathfrak{g} is not isomorphic to the semisimple Lie algebra of type $B_2 \oplus A_2 \oplus A_1^{\oplus 2}$.

Suppose that the assertion is false. Then the rank of \mathfrak{g} is equal to 6, which implies that \mathfrak{g}_0 is a Cartan subalgebra of \mathfrak{g} , and \mathfrak{g}_j^α is a root space of \mathfrak{g} for each $\alpha \in \mathfrak{g}_0^*$ and $j = 1, 2$

(note that they are all one-dimensional). Let L be the ideal of \mathfrak{g} isomorphic to B_2 , and let α_1, α_2 be the simple roots for $L \cong B_2$ such that $\theta := 2\alpha_1 + \alpha_2$ is the highest root of $L \cong B_2$. The root space L_θ is contained in \mathfrak{g}_1 or \mathfrak{g}_2 since it is one-dimensional. Here we assume that $L_\theta \subset \mathfrak{g}_1$; the proof for the case that $L_\theta \subset \mathfrak{g}_2$ is similar. If the root space $L_{-\alpha_1}$ is contained in \mathfrak{g}_2 , then we have

$$\{0\} \neq L_{\alpha_1+\alpha_2} = [L_\theta, L_{-\alpha_1}] \subset \mathfrak{g}_0$$

by (iii), which is a contradiction. If $L_{-\alpha_1} \subset \mathfrak{g}_1$, then we have

$$\{0\} \neq L_{\alpha_2} = [[L_\theta, L_{-\alpha_1}], L_{-\alpha_1}] \subset \mathfrak{g}_0$$

by (iii), which is also a contradiction. Thus we have proved Claim 2.

By Claims 1 and 2, along with (3.3.2), we conclude that \mathfrak{g} is isomorphic to the semisimple Lie algebra of type A_2^3 . Thus we have proved the proposition. \square

To complete a proof of Theorem 3.2.1, it remains to prove Proposition 3.3.3.

3.4 Proof of Proposition 3.3.3. By Theorem 3.1.1 (2), $\{w_\infty, w_1, w_3, w_4, w_5, w_9\}$ forms an \mathbb{F}_3 -basis of $\mathcal{C}_{12} = L/Q$. If we set $v_\infty := w_\infty$, and

$$v_i := -(w_i + w_\infty) = \sum_{j \in \Omega \setminus \nu^i \Theta} \overline{1}^{(j)} \quad \text{for } i = 1, 3, 4, 5, 9,$$

then $\{v_\infty, v_1, v_3, v_4, v_5, v_9\}$ is also an \mathbb{F}_3 -basis of \mathcal{C}_{12} . Define $u_j \in \mathcal{C}_{12}$ for $1 \leq j \leq 5$ by:

$$u_1 = v_1 + v_3 - v_4, \quad u_2 = v_1 + v_3 + v_4 + v_5 - v_9, \quad u_3 = -v_1 + v_3 - v_4 + v_5,$$

$$u_4 = v_\infty - v_1 - v_4 + v_5, \quad u_5 = -v_1 + v_3 + v_4 + v_5 + v_9, \quad u_6 := v_1$$

We can easily show the following lemma.

Lemma 3.4.1. *The set $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ is an \mathbb{F}_3 -basis of \mathcal{C}_{12} .*

For $c_i \in \mathbb{Z}$, $i \in \Omega$, we set

$$[c_\infty, c_4, c_7 \mid c_0, c_3, c_6 \mid c_1, c_5, c_8 \mid c_2, c_{10}, c_9] := \sum_{i \in \Omega} c_i [1]^{(i)} \in Q^*.$$

Define $t_j \in Q^*$ for $1 \leq j \leq 6$ by:

$$t_1 = [10\bar{1}|000|1\bar{1}0|\bar{1}10], \quad t_2 = [01\bar{1}|00\bar{1}|100|\bar{1}01], \quad t_3 = [000|011|011|0\bar{2}\bar{2}],$$

$$t_4 = [000|11\bar{1}|11\bar{1}|\bar{2}\bar{2}2], \quad t_5 = [000|000|11\bar{1}|\bar{1}\bar{1}1], \quad t_6 := [101|001|110|001]$$

where we set $\bar{n} := -n \in \mathbb{Z}$ for $n \in \mathbb{Z}$. The following lemma can be shown by direct computation.

Lemma 3.4.2. *For each $1 \leq j \leq 6$, the coset $u_j \in L/Q$ contains the element $t_j \in Q^*$. Therefore, t_j is contained in L for every $1 \leq j \leq 6$, and the lattice L is generated by $Q = A_2^{12}$ and t_j for $1 \leq j \leq 6$.*

Set $Q_0 := N \cap A_2^{12}$; notice that by (3.1.2),

$$Q_0 = \{(\alpha_1 \mid \alpha_2 \mid \alpha_3 \mid -\alpha_2 - \alpha_3) \in A_2^{12} \mid \alpha_1, \alpha_2, \alpha_3 \in A_2^3\}. \quad (3.4.1)$$

Lemma 3.4.3. *The lattice N is generated by Q_0 and t_j for $1 \leq j \leq 5$.*

Proof. Set $N' := \langle Q_0, t_j \mid 1 \leq j \leq 5 \rangle$. It can be easily seen that $t_j + \sigma t_j + \sigma^2 t_j = 0$ for every $1 \leq j \leq 5$. Hence, $N' \subset N$.

Let us show the reverse inclusion. We see from Lemma 3.4.2 that each element $\lambda \in L$ can be written as:

$$\lambda = \sum_{j=1}^6 a_j t_j + \beta \quad \text{with some } a_j \in \mathbb{Z} \text{ and } \beta \in A_2^{12}.$$

Assume that λ of the form above is contained in N . Since $t_j \in N'$ for every $1 \leq j \leq 5$ as seen above, λ is congruent to $a_6 t_6 + \beta$ modulo $N' \subset N$. In particular, $a_6 t_6 + \beta \in N$, and hence

$$a_6 \underbrace{(t_6 + \sigma t_6 + \sigma^2 t_6)}_{=[000|112|112|112]} + \underbrace{(\beta + \sigma\beta + \sigma^2\beta)}_{\in A_2^{12}} = 0.$$

Thus we get $a_6[000|112|112|112] \in A_2^{12}$, which implies that $a_6 \in 3\mathbb{Z}$. Consequently, $a_6 t_6 \in A_2^{12}$, and hence $a_6 t_6 + \beta \in N \cap A_2^{12} = Q_0 \subset N'$. Therefore we obtain $\lambda \in N'$, as desired. This completes the proof of the lemma. \square

Note that $3t_j \in Q_0$ for $1 \leq j \leq 5$. By Lemma 3.4.3, each element of N/Q_0 can be written as:

$$\sum_{j=1}^5 b_j(t_j + Q_0) \quad \text{with some } b_j \in \mathbb{F}_3, 1 \leq j \leq 5.$$

In addition, we deduce from Lemmas 3.4.1 and 3.4.2 that $\{t_j + Q_0 \mid 1 \leq j \leq 5\}$ is linearly independent over \mathbb{F}_3 . Thus,

$$|N/Q_0| = 3^5. \quad (3.4.2)$$

Lemma 3.4.4. (1) *The lattice M is generated by*

$$(1 - \sigma)A_2^{12} = \{(\mu \mid \alpha \mid \beta \mid -\alpha - \beta) \in A_2^{12} \mid \mu \in ((1 - \varphi)A_2)^3, \alpha, \beta \in A_2^3\}, \quad (3.4.3)$$

$$(1 - \sigma)t_1 = [000|1\bar{1}0|1\bar{1}0|\bar{2}20] + (1, 0, -1)^{(1)} - (1, 0, -1)^{(3)},$$

$$(1 - \sigma)t_2 = [000|10\bar{2}|101|\bar{2}01] + (1, 0, -1)^{(2)} - (1, 0, -1)^{(3)},$$

$$(1 - \sigma)t_5 = [000|11\bar{1}|11\bar{1}|\bar{2}\bar{2}2],$$

$$(1 - \sigma)t_6 = [000|000|11\bar{1}|\bar{1}\bar{1}1] + (1, 0, -1)^{(1)} + (1, 0, -1)^{(3)}.$$

(2) *We have $|Q_0/(1 - \sigma)A_2^{12}| = 3^3$ and $|M/(1 - \sigma)A_2^{12}| = 3^4$.*

Proof. (1) By Lemma 3.4.2, M is generated by $(1 - \sigma)A_2^{12}$ and $(1 - \sigma)t_j$ for $1 \leq j \leq 6$. We can easily check that $(1 - \sigma)t_3, (1 - \sigma)t_4 \in (1 - \sigma)A_2^{12}$. Therefore, M is generated by $(1 - \sigma)A_2^{12}$ and $(1 - \sigma)t_j$ for $j = 1, 2, 5, 6$.

(2) By (3.4.1) and (3.4.3), $|Q_0/(1 - \sigma)A_2^{12}| = |A_2/(1 - \varphi)A_2|^3$. We see that

$$\{(1 - \varphi)(0, 1, -1) = (1, 1, -2), (1 - \varphi)(1, 0, -1) = (2, -1, -1)\}$$

is a basis of $(1 - \sigma)A_2$. Thus, $\{(3, 0, -3), (2, -1, -1)\}$ is also a basis of $(1 - \varphi)A_2$. Since $\{(1, 0, -1), (2, -1, -1)\}$ is a basis of A_2 , we get $|A_2/(1 - \varphi)A_2| = 3$, and hence $|Q_0/(1 - \sigma)A_2^{12}| = 3^3$.

Next, it is obvious that $3(1 - \sigma)t_j \in (1 - \sigma)A_2^{12}$ for every $j = 1, 2, 5, 6$. Thus each element of $M/(1 - \sigma)A_2^{12}$ can be written as an \mathbb{F}_3 -linear combination of $(1 - \sigma)t_j + (1 - \sigma)A_2^{12}$, $j = 1, 2, 5, 6$. Assume that

$$c_1(1 - \sigma)t_1 + c_2(1 - \sigma)t_2 + c_5(1 - \sigma)t_5 + c_6(1 - \sigma)t_6 \in (1 - \sigma)A_2^{12}$$

for some $c_1, c_2, c_5, c_6 \in \mathbb{F}_3$. Then we have

$$\begin{cases} (c_1 + c_6)(1, 0, -1) \in (1 - \varphi)A_2, \\ c_2(1, 0, -1) \in (1 - \varphi)A_2, \\ (-c_1 - c_2 + c_6)(1, 0, -1) \in (1 - \varphi)A_2. \end{cases}$$

Because $\{(3, 0, -3), (2, -1, -1)\}$ is a basis of $(1 - \varphi)A_2$, it follows that $c_2 = 0 \in \mathbb{F}_3$. Similarly, we get $c_1 + c_6 = 0 \in \mathbb{F}_3$ and $-c_1 + c_6 = 0 \in \mathbb{F}_3$, which implies that $c_1 = c_6 = 0 \in \mathbb{F}_3$. Thus, $(1 - \sigma)t_j + (1 - \sigma)A_2^{12}$, $j = 1, 2, 5, 6$, are linearly independent over \mathbb{F}_3 . Therefore, $|M/(1 - \sigma)A_2^{12}| = 3^4$. This completes the proof of the lemma. \square

Now, let us show Proposition 3.3.3, that is, $|N/R| = |N/M| = 3^4$. Since $|N/M| \cdot |M/(1 - \sigma)A_2^{12}| = |N/Q_0| \cdot |Q_0/(1 - \sigma)A_2^{12}|$, it follows from (3.4.2) and Lemma 3.4.4 (2) that

$$|N/M| = \frac{|N/Q_0| \cdot |Q_0/(1 - \sigma)A_2^{12}|}{|M/(1 - \sigma)A_2^{12}|} = 3^4.$$

Thus we have proved the proposition.

4 Automorphism by which the fixed point lattices are of rank 6.

4.1 Main result in §4. Let L be a Niemeier lattice, and let τ be an arbitrary lattice automorphism of L of order 3. In this section, we show the following theorem; we should remark (cf. (3.1.1)) that in the theorem below,

$$\text{rank } L^\tau = 6 \iff \rho = \frac{1}{18}(\dim \mathfrak{h}_{(1)} + \dim \mathfrak{h}_{(2)}) = 1,$$

where L^τ denotes the fixed point sublattice of L under τ .

Theorem 4.1.1. *Let L be a Niemeier lattice, and let τ be an arbitrary lattice automorphism of L of order 3 such that $\text{rank } L^\tau = 6$. The holomorphic VOA \tilde{V}_L^τ obtained by applying Theorem 2.3.2 to L and τ is isomorphic to the lattice VOA associated to a Niemeier lattice, the one obtained in Theorem 3.2.1, which corresponds to No. 6 on Schellekens' list [S, Table 1], or the one obtained in [Mi, §5.2], which corresponds to No. 32 on Schellekens' list.*

We prove this theorem as follows: Let Q be the root lattice for L , and $W = W(Q)$ the Weyl group for Q ; recall that W is a normal subgroup of $\text{Aut}(L)$. In §4.2, we show that if τ is contained in W , then \tilde{V}_L^τ is isomorphic to the lattice VOA associated to a Niemeier lattice. In §4.3, we prove that if $L = \Lambda$, the Leech lattice, then \tilde{V}_L^τ is isomorphic to the Leech lattice VOA V_Λ itself. In §4.4, we consider the case that L is not isomorphic to Λ , and τ is not contained in W . In Proposition 4.4.2, we first prove that the root lattice Q is isomorphic to either A_2^{12} or E_6^4 . Finally, in Proposition 4.4.4, we prove that if $Q = A_2^{12}$ (resp., $Q = E_6^4$), then the automorphism τ is conjugate, in $\text{Aut}(L)$, to the one given in §3.1 (resp., in [Mi, §5.2]), and hence \tilde{V}_L^τ is isomorphic to the one obtained in Theorem 3.2.1 (resp., [Mi, §5.2]).

4.2 Automorphisms that are inner on the weight one spaces. We first assume more generally that L is a positive-definite, even lattice. Let V_L be the lattice VOA associated to L , with $a \in V_L \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in (\text{End}_{\mathbb{C}} V_L)[[z, z^{-1}]]$ the vertex operator. For each $a \in (V_L)_1$, $\exp a_0$ is a VOA automorphism of V_L (see [DN, §2.3]). Set

$$G := \langle \exp a_0 \mid a \in (V_L)_1 \rangle \subset \text{Aut}(V_L).$$

Note that the restriction of an element in G to $(V_L)_1$ is an inner automorphism of the Lie algebra $(V_L)_1$.

Lemma 4.2.1. *Keep the notation and setting above. Let $\tau \in G$ be of finite order. Then the fixed point subVOA V_L^τ of V_L under τ is isomorphic to a lattice VOA.*

Proof. First, we note that $(V_L)_1$ is reductive. By [K, Proposition 8.1], there exists a Cartan subalgebra \mathfrak{h}' of $(V_L)_1$ such that $\tau = \exp h_0$ for some $h \in \mathfrak{h}'$. Since Cartan subalgebras of $(V_L)_1$ are conjugate under G , there exists $g \in G$ such that $g(\mathfrak{h}')$ is identical to the canonical Cartan subalgebra $\{h(-1)1 \otimes e^0 \mid h \in \mathfrak{h}\}$ of $(V_L)_1$. Set $\tau' = g\tau g^{-1} = \exp g(h)_0$. Then, τ' acts on $M(1) \otimes e^\beta$ as the scalar multiple by $\exp \langle g(h), \beta \rangle$ for each $\beta \in L$. Denote by $|\tau|$ the order of τ , which is finite by assumption, and set $v := |\tau|g(h)/2\pi\sqrt{-1}$. Then, $\exp \langle g(h), \beta \rangle = \exp(2\pi\sqrt{-1}\langle v, \beta \rangle/|\tau|)$ for $\beta \in L$. Thus we see that $\langle v, \beta \rangle \in \mathbb{Z}$ for every $\beta \in L$, and that τ' acts trivially on $M(1) \otimes e^\beta$ if and only if $\langle v, \beta \rangle \in |\tau|\mathbb{Z}$. So, let us set $K := \{\beta \in L \mid \langle v, \beta \rangle \in |\tau|\mathbb{Z}\} \subset L$. Since $\langle v, L \rangle \subset \mathbb{Z}$ as seen above, we have $|\tau|L \subset K$, and hence $K \otimes_{\mathbb{Z}} \mathbb{C} = L \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}$. Therefore we conclude that $V_L^{\tau'} = V_K$. Since τ' is conjugate to τ by definition, it follows immediately that $V_L^\tau \cong V_L^{\tau'}$. Combining these, we obtain $V_L^\tau \cong V_K$, thereby completing the proof of the lemma. \square

Lemma 4.2.2. *Let K be a positive-definite, even lattice. If U is a simple current extension of the lattice VOA V_K , then U is isomorphic to the lattice VOA associated to a sublattice of K^* .*

Proof. By [D], an irreducible V_K -module is isomorphic to $V_{\lambda+K}$ for some $\lambda + K \in K^*/K$. Hence there exists a subset $S \subset K^*/K$ such that $U \cong \bigoplus_{\lambda+K \in S} V_{\lambda+K}$. Since $V_{\lambda+K} \boxtimes V_{\mu+K} \cong V_{\lambda+\mu+K}$ (see [DL1, Corollary 12.10]), it follows immediately that the subset S is a subgroup of K^*/K . Hence there exists a sublattice $T \subset K^*$ such that $S = T/K$. By the uniqueness of simple current extensions (see [DM2, Proposition 5.3]), we have $U \cong V_T$ as VOAs. Thus we have proved the lemma. \square

Combining these lemmas, we obtain the following proposition.

Proposition 4.2.3. *Let L be a positive-definite, even lattice, and let $\tau \in G$ be of finite order. A simple current extension of the fixed point subVOA V_L^τ is isomorphic to a lattice VOA.*

Keep the setting in Theorem 4.1.1, and assume that $\tau \in W$. We deduce from [K, Lemma 3.8] and [DN, Lemma 2.5] that the VOA automorphism of V_L induced from τ is contained in $G = \langle \exp a_0 \mid a \in (V_L)_1 \rangle$. Thus it follows from Remark 2.3.3 and Proposition 4.2.3 that the VOA \tilde{V}_L^τ is isomorphic to a lattice VOA. Because the central charge of \tilde{V}_L^τ is equal to 24, the rank of the lattice is equal to 24. Also, because \tilde{V}_L^τ is holomorphic, the lattice is unimodular. Hence the lattice is a Niemeier lattice. Thus we have proved that if $\tau \in W$, then \tilde{V}_L^τ is isomorphic to the lattice VOA associated to a Niemeier lattice.

4.3 Case of the Leech lattice. Let Λ denote the Leech lattice. Note that Λ is a unique Niemeier lattice of rank 24 without roots (or equivalently, $Q = \{0\}$).

Proposition 4.3.1. *Let τ be a lattice automorphism of Λ of order 3 such that $\text{rank } \Lambda^\tau = 6$. Then the VOA \tilde{V}_Λ^τ is isomorphic to V_Λ .*

Proof. We see that $(V_\Lambda^\tau)_1 = \{h(-1)1 \otimes e^0 \mid h \in \mathfrak{h}_{(0)}\}$, and hence is a 6-dimensional abelian Lie subalgebra. By Lemma 2.2.2 (1), $(V_\Lambda^\tau)_1$ is an ideal of $(\tilde{V}_\Lambda^\tau)_1$, which implies that $(\tilde{V}_\Lambda^\tau)_1$ is not semisimple. Therefore, by [DM1, Theorem 3], we conclude that $\tilde{V}_\Lambda^\tau \cong V_\Lambda$, as desired. \square

4.4 Automorphisms not contained in the Weyl group. Keep the setting in Theorem 4.1.1. By the arguments in §4.2 and §4.3, we may assume that L is not isomorphic to the Leech lattice Λ (and hence $Q \neq \{0\}$), and $\tau \notin W$. We start with the following preliminary lemma.

Lemma 4.4.1. (1) *If ψ is a lattice automorphism of the root lattice A_n , $n \geq 2$, of order 3, then $\text{rank } A_n^\psi = n - 2c$ for some $1 \leq c \leq (n+1)/3$.*

(2) *If ψ is a lattice automorphism of the root lattice D_n , $n \geq 4$, of order 3, then $\text{rank } D_n^\psi = n - 2c$ for some $1 \leq c \leq n/3$.*

Proof. (1) Recall that $\text{Aut}(A_n) = 2 \times \mathfrak{S}_{n+1}$, where \mathfrak{S}_{n+1} is the symmetric group of degree $n+1$. Then, ψ is a product of mutually commutative 3-cycles, and the number c of 3-cycles in ψ satisfies $1 \leq c \leq (n+1)/3$. It is easy to see that the rank of A_n^ψ is equal to $n - 2c$.

(2) Recall that $\text{Aut}(D_n) \cong 2^n : \mathfrak{S}_n$ if $n > 4$, and $\text{Aut}(D_4) \cong 2^3 : \mathfrak{S}_4 : \mathfrak{S}_3$. If $\psi \in 2^n : \mathfrak{S}_n$, then ψ is a product of mutually commutative conjugations of 3-cycles by 2^n (i.e., mutually commutative “signed” 3-cycles). As in the case of A_n , we see that the number c of them in ψ satisfies $1 \leq c \leq n/3$, and the rank of D_n^ψ is equal to $n - 2c$. Also, it can be easily checked that the rank of D_4^ψ is equal to 2 when $n = 4$ and ψ contains a Dynkin diagram automorphism of order 3. Thus we have proved the lemma. \square

Let $Q = Q_1 \oplus \cdots \oplus Q_n$ be the (orthogonal) decomposition of Q into indecomposable root lattices, and let $W(Q_m)$ be the Weyl group for Q_m , $1 \leq m \leq n$; note that $W = W(Q) = W(Q_1) \times \cdots \times W(Q_n)$.

Proposition 4.4.2. *Keep the setting in Theorem 4.1.1. Assume that $L \neq \Lambda$ and $\tau \notin W$.*

- (1) *For some $1 \leq m \leq n$, $\tau Q_m \not\subset Q_m$.*
- (2) *The root lattice Q is isomorphic to either E_6^4 or A_2^{12} .*

Proof. (1) Suppose that $\tau(Q_m) \subset Q_m$ for all $1 \leq m \leq n$. Let us denote by τ_m the restriction of τ to Q_m . Then, τ_m is an automorphism of Q_m . It follows from, e.g., [K, Corollary 5.10 (b)] that $\tau_m = d_m w_m$ with some Dynkin diagram automorphism d_m for Q_m and $w_m \in W(Q_m)$. Since $\tau \notin W$ by assumption, $d_m \notin W(Q_m)$ for some $1 \leq m \leq n$. Then, d_m is of order 3 (since so is τ), and hence $Q_m = D_4$. Thus it follows from [CS, Chapter 16, Table 16.1] that $Q = A_5^4 D_4$ or $Q = D_4^6$. If $Q = A_5^4 D_4$, then $|G_1| = 2$ with notation in [CS, Chapter 16, Table 16.1], which implies that there is no element in $\text{Aut}(L)$ of order 3 that is not contained in W , and preserves every indecomposable component in Q (such as τ). So, we get $Q = D_4^6$. However, we deduce by Lemma 4.4.1 that $\text{rank } Q_m^{\tau_m} = 2$ for each $1 \leq m \leq 6$, and hence $\text{rank } L^\tau \geq \text{rank } Q^\tau = 2 \times 6 = 12 > 6$, which contradicts our assumption. Thus we have proved part (1).

(2) By part (1), τ acts nontrivially on the set $\{Q_m \mid 1 \leq m \leq n\}$ of indecomposable components in Q . We may assume that $\tau Q_{3l+1} \subset Q_{3l+2}$ and $\tau Q_{3l+2} \subset Q_{3l+3}$ for each $0 \leq l \leq k-1$ and $\tau Q_m \subset Q_m$ for every $3k+1 \leq m \leq n$, where $1 \leq k \leq n/3$; we should remark that $Q_{3l+1} \cong Q_{3l+2} \cong Q_{3l+3}$ for each $0 \leq l \leq k-1$. For each $0 \leq l \leq k-1$, we set

$$\tilde{Q}_l := \{\alpha + \tau\alpha + \tau^2\alpha \mid \alpha \in Q_{3l+1}\} = (Q_{3l+1} \oplus Q_{3l+2} \oplus Q_{3l+3})^\tau \subset Q^\tau;$$

notice that $\text{rank } \tilde{Q}_l = \text{rank } Q_{3l+1}$. Then, we have

$$\text{rank } L^\tau \geq \text{rank } Q^\tau = \sum_{l=0}^{k-1} \text{rank } \tilde{Q}_l + \sum_{m=3k+1}^n \text{rank } Q_m^\tau = \sum_{l=0}^{k-1} \text{rank } Q_{3l+1} + \sum_{m=3k+1}^n \text{rank } Q_m^\tau \quad (4.4.1)$$

Now, suppose that Q is isomorphic to neither E_6^4 nor A_2^{12} . Because $k \geq 1$ and $Q_{3l+1} \cong Q_{3l+2} \cong Q_{3l+3}$ for each $0 \leq l \leq k-1$, it follows from [CS, Chapter 16, Table 16.1] that Q is isomorphic to one of

$$A_1^{24}, \quad A_3^8, \quad A_4^6, \quad A_5^4 D_4, \quad D_4^6, \quad A_6^4, \quad D_6^4, \quad A_8^3, \quad D_8^3, \quad E_8^3.$$

Suppose that $Q \cong A_8^3, D_8^3$, or E_8^3 ; note that $n = 3$ and $k = 1$ in these cases. Then we see from (4.4.1) that $\text{rank } L^\tau \geq \text{rank } Q^\tau = 8$, which contradicts the assumption that $\text{rank } L^\tau = 6$.

If $Q = A_1^{24}$, then $n = 24$ and $1 \leq k \leq 8$. Since $\text{Aut } A_1 \cong \mathbb{Z}/2\mathbb{Z}$, and hence does not contain an element of order 3, it follows that $Q_m^\tau = A_1$ for all $3k+1 \leq m \leq n$. Thus, by (4.4.1), $\text{rank } L^\tau \geq k \cdot \text{rank } A_1 + (24-3k) \cdot \text{rank } A_1 = 24-2k \geq 8$, which is a contradiction.

If $Q = A_3^8$, then $n = 8$ and $1 \leq k \leq 2$. By Lemma 4.4.1 (1), $\text{rank } Q_m^\tau \geq 3-2 \cdot (3+1)/3$, and hence $\text{rank } Q_m^\tau \geq 1$. Thus, $\text{rank } L^\tau \geq k \cdot \text{rank } A_3 + (8-3k) \cdot 1 = 8$, which is a contradiction. In exactly the same way as this, we can obtain the contradiction that $\text{rank } L^\tau > 6$ also in the cases of $Q = A_4^6, A_5^4 D_4, D_4^6, A_6^4$, and D_6^4 . Thus we have proved the proposition. \square

In the rest of this subsection, we assume that the root lattice Q of the Niemeier lattice L is isomorphic to either A_2^{12} or E_6^4 . Define

$$P := \begin{cases} A_2^3 & \text{if } Q = A_2^{12}, \\ E_6 & \text{if } Q = E_6^4; \end{cases}$$

note that $Q = P \oplus P \oplus P \oplus P$. When $Q \cong A_2^{12}$ (resp., $Q \cong E_6^4$), let $\sigma \in \text{Aut}(L)$ be the one given in §3.1 (resp., in [Mi, §5.2]; see also §A.1 in Appendix below). Recall that σ maps $(\mu_1, \mu_2, \mu_3, \mu_4) \in Q^* = P^* \oplus P^* \oplus P^* \oplus P^*$ to $(x\mu_1, \mu_4, \mu_2, \mu_3) \in Q^*$, where x is an element of $W(P)$ that acts fixed-point-freely on P .

Lemma 4.4.3. *Keep the notation and setting above. Let $w = (1, w_2, w_3, w_4) \in W(Q)$, with $w_m \in W(P)$, $2 \leq m \leq 4$, and assume that the order of $w\sigma$ is equal to 3. Then, $w\sigma$ is conjugate to σ in $\text{Aut}(L)$.*

Proof. Since $1 = (w\sigma)^3 = (1, w_2 w_4 w_3, w_3 w_2 w_4, w_4 w_3 w_2)$, we have $w_4 = w_2^{-1} w_3^{-1}$. Then we see by simple computation that $w\sigma g = g\sigma$ with $g := (1, w_2, w_3 w_2, 1) \in W(Q)$. Thus we have proved the lemma. \square

It holds that $\text{Aut}(L) \cong W(Q) : H$ with $H := \text{Aut}(L/Q)$; note that

$$H \cong \begin{cases} 2.M_{12} & \text{if } Q \cong A_2^{12}, \\ 2.\mathfrak{S}_4 & \text{if } Q \cong E_6^4, \end{cases}$$

where M_{12} denotes the Mathieu group of degree 12, and the center $Z(H)$ of H is generated by $-1 \in \text{Aut}(L)$. Define $\sigma_W, \sigma_H \in \text{Aut}(Q^*)$ by:

$$\sigma_H : (\mu_1, \mu_2, \mu_3, \mu_4) \mapsto (\mu_1, \mu_4, \mu_2, \mu_3),$$

$$\sigma_W : (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3, \boldsymbol{\mu}_4) \mapsto (x\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3, \boldsymbol{\mu}_4)$$

for $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3, \boldsymbol{\mu}_4 \in P^*$. We deduce that $\sigma_W \in W$, $\sigma_H \in H$, and $\sigma = \sigma_W \sigma_H$.

Proposition 4.4.4. *Assume that Q is isomorphic to either A_2^{12} or E_6^4 , and let σ be as above. If $\tau \in \text{Aut}(L)$ is a lattice automorphism of L of order 3 that is not contained in W , and satisfies the condition that $\text{rank } L^\tau = 6$, then τ is conjugate to σ in $\text{Aut}(L)$.*

Proof. Let $\tau_W \in W(Q)$ and $\tau_H \in H$ be such that $\tau = \tau_W \tau_H$; note that $\tau_H \neq 1$ since $\tau \notin W$. Since $W(Q)$ is a normal subgroup of $\text{Aut}(L)$, we see that the order of τ_H is equal to 3. Remark that τ_H is a unique element of order 3 in the coset $\tau_H Z(H) \in H/Z(H)$.

(i) If $Q = A_2^{12}$, then $H/Z(H) \cong M_{12}$. We deduce from the character table of M_{12} (see [ATLAS]) that H has exactly two conjugacy classes of elements of order 3; the one acts fixed-point-freely on the set $\{Q_m \cong A_2 \mid 1 \leq m \leq 12\}$ of indecomposable components in Q , and the other fixes exactly three indecomposable components; we should remark that σ_H is contained in the latter class. Suppose that τ_H is contained in the former class. Then, τ also acts fixed-point-freely on $\{Q_m \cong A_2 \mid 1 \leq m \leq 12\}$ since $\tau_W Q_m \subset Q_m$ for every $1 \leq m \leq 12$. Thus it follows from (4.4.1) that $\text{rank } L^\tau \geq 4 \cdot \text{rank } A_2 = 8$, which contradicts the assumption that $\text{rank } L^\tau = 6$. Hence, τ_H is conjugate to σ_H in H .

(ii) If $Q = E_6^4$, then $H/Z(H) \cong \mathfrak{S}_4$. By the uniqueness of a conjugacy class of \mathfrak{S}_4 containing an element of order 3, we see that τ_H is conjugate to σ_H in H .

We see by (i) and (ii) that $\tau = \tau_W \tau_H$ is conjugate to $w\sigma_H$ for some $w \in W(Q)$. So we may assume from the beginning that $\tau_H = \sigma_H$. Write τ_W as: $\tau_W = (w_1, w_2, w_3, w_4)$ with $w_m \in W(P)$, $1 \leq m \leq 4$. Because $\text{rank } L^\tau = 6$, it follows from (4.4.1) that $w_1 \in W(P)$ acts fixed-point-freely on P . Thus, w_1 is conjugate to x in $W(P)$ (for $P = E_6$, see the character table of $W(E_6) \cong U_4(2).2$ in [ATLAS]); let $g \in W(P)$ be such that $w_1 = gxg^{-1}$. Then,

$$\begin{aligned} \tau &= (w_1, w_2, w_3, w_4)\sigma_H = (1, w_2, w_3, w_4)(g, 1, 1, 1)(x, 1, 1, 1)(g^{-1}, 1, 1, 1)\sigma_H \\ &= (g, 1, 1, 1)(1, w_2, w_3, w_4) \underbrace{(x, 1, 1, 1)\sigma_H(g^{-1}, 1, 1, 1)}_{=\sigma}, \end{aligned}$$

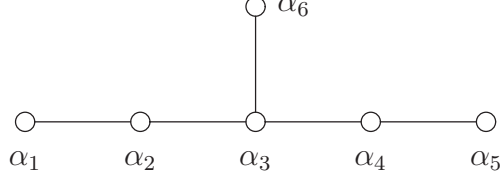
and hence τ is conjugate to $w\sigma$ with $w = (1, w_2, w_3, w_4)$. Because $w\sigma$ is conjugate to σ by Lemma 4.4.3, we conclude that τ is conjugate to σ . This completes the proof of the proposition. \square

A Appendix.

A.1 Holomorphic VOA corresponding to No.32 on Schellekens' list. In [Mi, §5.2], Miyamoto proved that the VOA obtained by applying Theorem 2.3.2 to the Niemeier lattice with E_6^4 the root lattice and a specified automorphism of order 3 corresponds to No.32 on Schellekens' list [S]. In the proof, he first showed, by using some fusion rules and

modular transformations for trace functions, that the dimension of the weight one space of the VOA is equal to 120. In this subsection, we give another proof for this fact by using the Dong-Lepowsky construction of twisted modules (see §2.2).

We first recall the definitions of $L = \text{Ni}(E_6^4)$ and the automorphism $\sigma \in \text{Aut}(L)$ given in [Mi, §5.2]. Let $\{\alpha_i \mid 1 \leq i \leq 6\}$ be the simple roots for E_6 ;



We set

$$[0] := 0, \quad [1] := \frac{1}{3}(\alpha_1 - \alpha_2 + \alpha_4 - \alpha_5), \quad [2] := \frac{1}{3}(-\alpha_1 + \alpha_2 - \alpha_4 + \alpha_5),$$

and

$$\overline{[0]} := [0] + E_6, \quad \overline{[1]} := [1] + E_6, \quad \overline{[2]} := [2] + E_6.$$

Then, $E_6^*/E_6 = \{\overline{[\ell]} \mid \ell = 0, 1, 2\}$, and the additive group E_6^*/E_6 is naturally isomorphic to $\mathbb{F}_3 := \mathbb{Z}/3\mathbb{Z}$. Define Q to be the direct sum E_6^4 of four copies of E_6 . Since $Q^* = (E_6^*)^4$, it follows that

$$Q^*/Q = \{(\overline{[\ell_1]}, \overline{[\ell_2]}, \overline{[\ell_3]}, \overline{[\ell_4]}) \mid \ell_i = 0, 1, 2 \text{ for each } 1 \leq i \leq 4\},$$

For $\overline{[\ell]} \in E_6^*/E_6$ and $i \in \Omega$, define $\overline{[\ell]}^{(i)}$ to be the element $(\overline{[\ell_1]}, \overline{[\ell_2]}, \overline{[\ell_3]}, \overline{[\ell_4]}) \in Q^*/Q$ with $\ell_i = \ell$ and $\ell_j = 0$ for all $1 \leq j \leq 4, j \neq i$. If we regard Q^*/Q as a 4-dimensional vector space over \mathbb{F}_3 , then $\{\overline{[1]}^{(i)} \mid i \in \Omega\}$ forms a basis of Q^*/Q , which we identify with the standard orthonormal basis of \mathbb{F}_3^4 . Set

$$v_1 := \overline{[1]}^{(2)} + \overline{[1]}^{(3)} + \overline{[1]}^{(4)}, \quad v_2 := \overline{[1]}^{(1)} + \overline{[1]}^{(2)} - \overline{[1]}^{(3)},$$

and define \mathcal{C}_4 to be the subspace of Q^*/Q spanned by v_1 and v_2 . Then, \mathcal{C}_4 is isomorphic to the tetracode, a unique self-dual code in \mathbb{F}_3^4 (see [CS, Chapter 3, §2.5.1]). Note that

$$\overline{[1]}^{(1)} + \overline{[1]}^{(3)} - \overline{[1]}^{(4)} = v_2 - v_1 \in \mathcal{C}_4. \tag{A.1.1}$$

We now set (see [CS, Chapter 18, §4, XIV])

$$(Q \subset) \quad \text{Ni}(E_6^4) := \bigsqcup_{C \in \mathcal{C}_4} C \quad (\subset Q^*).$$

Now, we first define an automorphism σ_0 of Q^* of order 3 by

$$(\mu_1, \mu_2, \mu_3, \mu_4) \xrightarrow{\sigma_0} (\mu_1, \mu_4, \mu_2, \mu_3)$$

for $\mu_1, \mu_2, \mu_3, \mu_4 \in E_6^*$. Then we deduce from (A.1.1) that σ_0 stabilizes the Niemeier lattice $\text{Ni}(E_6^4)$, and hence $\sigma_0 \in \text{Aut}(\text{Ni}(E_6^4))$.

Next, let φ denote the automorphism of E_6^* defined by: $\varphi = r_1 r_2 r_4 r_5 r_6 r_0$, where $r_i \in W(E_6)$ is the reflection with respect to the simple root α_i for $1 \leq i \leq 6$, and $r_0 \in W(E_6)$ is the reflection with respect to the highest root of E_6 ; note that φ acts fixed-point-freely on $E_6 \otimes \mathbb{C}$, but $\varphi(\overline{[\ell]}) = \overline{[\ell]}$ for every $\ell = 0, 1, 2$. So, if we define an automorphism σ_1 of Q^* (of order 3) by:

$$(\mu_1, \mu_2, \mu_3, \mu_4) \xrightarrow{\sigma_1} (\varphi(\mu_1), \mu_2, \mu_3, \mu_4)$$

for $\mu_1, \mu_2, \mu_3, \mu_4 \in E_6^*$, then σ_1 stabilizes the Niemeier lattice $\text{Ni}(E_6^4)$, and hence $\sigma_1 \in \text{Aut}(\text{Ni}(E_6^4))$. The automorphism σ given in [Mi, §5.2] is identical to $\sigma_0 \circ \sigma_1$.

The fixed point subspace $\mathfrak{h}_{(0)}$ of $\mathfrak{h} = \text{Ni}(E_6^4) \otimes \mathbb{C}$ under the σ above is identical to $\{(0, h, h, h) \mid h \in E_6 \otimes \mathbb{C}\}$, and hence $\dim \mathfrak{h}_{(0)} = 6$. Also, remark that

$$\mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)} = \{(h, h_1, h_2, -h_1 - h_2) \mid h, h_1, h_2 \in E_6 \otimes \mathbb{C}\}. \quad (\text{A.1.2})$$

Now, since $\dim \mathfrak{h}_{(0)} = 6$, the ρ defined by (2.2.6) is equal to $1 \in (1/3)\mathbb{Z}$. Therefore we can apply Theorem 2.3.2 to $L = \text{Ni}(E_6^4)$ and this σ , and obtain a holomorphic VOA \tilde{V}_L^σ . We will show that

$$\dim(\tilde{V}_L^\sigma)_1 = 120. \quad (\text{A.1.3})$$

By definition, $(\tilde{V}_L^\sigma)_1 = (V_L^\sigma)_1 \oplus V(\sigma)_1 \oplus V(\sigma^2)_1$. It can be easily checked that $\dim(V_L^\sigma)_1 = 112$. Also, $\dim V(\sigma)_1 = \dim V(\sigma^2)_1$ by Remark 2.3.1. Thus it suffices to show that

$$\dim V(\sigma)_1 = 9 \quad (\text{A.1.4})$$

Define sublattices N, R, M of $L = \text{Ni}(E_6^4)$ as (2.2.2) and (2.2.3); recall that $M = R$ by Proposition 2.4.1. Because $\rho = 1$, it follows immediately from (2.2.7) that $\dim V(\sigma)_1 = |N/R|^{1/2} = |N/M|^{1/2}$. Hence equality (A.1.4) follows immediately from the next proposition.

Proposition A.1.1. *We have $|N/R| = |N/M| = 3^4$.*

In order to prove this proposition, we need some lemmas. For $c_i \in \mathbb{Z}$, $1 \leq i \leq 4$, we set

$$[c_1, c_2, c_3, c_4] := (c_1[1], c_2[1], c_3[1], c_4[1]) \in Q^*.$$

Define $t_1, t_2 \in Q^*$ by: $t_1 = [0, 1, 1, -2]$ and $t_2 = [1, 1, -1, 0]$. The following lemma can be shown by direct computation.

Lemma A.1.2. *For each $1 \leq j \leq 2$, the coset $v_j \in L/Q$ contains the element $t_j \in Q^*$. Therefore, t_1 and t_2 are contained in L , and the lattice L is generated by $Q = E_6^4$ and t_1, t_2 .*

Set $Q_0 := N \cap E_6^4$; notice that by (A.1.2),

$$Q_0 = \{(\alpha_1, \alpha_2, \alpha_3, -\alpha_2 - \alpha_3) \in E_6^4 \mid \alpha_1, \alpha_2, \alpha_3 \in E_6\}. \quad (\text{A.1.5})$$

Lemma A.1.3. *The lattice N is generated by Q_0 and t_1, t_2 .*

Proof. Set $N' := \langle Q_0, t_j \mid 1 \leq j \leq 2 \rangle$. It can be easily seen that $t_j + \sigma t_j + \sigma^2 t_j = 0$ for every $1 \leq j \leq 2$. Hence, $N' \subset N$.

Let us show the reverse inclusion. We see that each element $\lambda \in L$ can be written as:

$$\lambda = \sum_{j=1}^2 a_j t_j + \beta \quad \text{with some } a_j \in \mathbb{Z} \text{ and } \beta \in E_6^4.$$

Assume that λ of the form above is contained in N . Since $t_j \in N'$ for every $1 \leq j \leq 2$ as seen above, λ is congruent to β modulo $N' \subset N$. Hence, $\beta \in N \cap E_6^4 = Q_0 \subset N'$. Therefore we obtain $\lambda \in N'$, as desired. This completes the proof of the lemma. \square

Note that $3t_j \in Q_0$ for $1 \leq j \leq 2$. By Lemma A.1.3, each element of N/Q_0 can be written as:

$$\sum_{j=1}^2 b_j(t_j + Q_0) \quad \text{with some } b_j \in \mathbb{F}_3, 1 \leq j \leq 2.$$

In addition, we deduce that $\{t_j + Q_0 \mid 1 \leq j \leq 2\}$ is linearly independent over \mathbb{F}_3 . Thus,

$$|N/Q_0| = 3^2. \tag{A.1.6}$$

Lemma A.1.4. (1) *The lattice M is generated by $(1 - \sigma)t_2$ and*

$$(1 - \sigma)E_6^4 = \{(\mu, \alpha, \beta, -\alpha - \beta) \in E_6^4 \mid \mu \in (1 - \varphi)E_6, \alpha, \beta \in E_6\}. \tag{A.1.7}$$

(2) *We have $|Q_0/(1 - \sigma)E_6^4| = 3^3$ and $|M/(1 - \sigma)E_6^4| = 3$.*

Proof. (1) By Lemma A.1.2, M is generated by $(1 - \sigma)E_6^4$ and $(1 - \sigma)t_j$ for $1 \leq j \leq 2$. We can easily check that $(1 - \sigma)t_1 \in (1 - \sigma)E_6^4$. Therefore, M is generated by $(1 - \sigma)E_6^4$ and $(1 - \sigma)t_2$.

(2) By (A.1.5) and (A.1.7), $|Q_0/(1 - \sigma)E_6^4| = |E_6/(1 - \varphi)E_6|$. The same argument as Claim 3 in the proof of Proposition 2.4.1 (with N and σ replaced by E_6 and φ , respectively) shows that $|E_6/(1 - \varphi)E_6| = 3^{(\text{rank } E_6)/2} = 3^3$. Thus we have proved $|Q_0/(1 - \sigma)E_6^4| = 3^3$.

Next, it is obvious that $3(1 - \sigma)t_2 \in (1 - \sigma)E_6^4$ and $(1 - \sigma)t_2 \notin (1 - \sigma)E_6^4$. Hence $|M/(1 - \sigma)E_6^4| = 3$. This completes the proof of the lemma. \square

Proof of Proposition A.1.1. Since $|N/M| \cdot |M/(1 - \sigma)E_6^4| = |N/Q_0| \cdot |Q_0/(1 - \sigma)E_6^4|$, it follows from (A.1.6) and Lemma A.1.4 (2) that

$$|N/M| = \frac{|N/Q_0| \cdot |Q_0/(1 - \sigma)E_6^4|}{|M/(1 - \sigma)E_6^4|} = 3^4.$$

Thus we have proved Proposition A.1.1. \square

As in Section 3.3, we see that $(V_L^\sigma)_1 = \mathfrak{g}^{(234)} \oplus \mathfrak{s}$, where $\mathfrak{g}^{(234)}$ is isomorphic to the simple Lie algebra of type E_6 , and \mathfrak{s} is isomorphic to the semisimple Lie algebra of type A_2^3 .

Proposition A.1.5. *Keep the notation and setting above. The Lie algebra $(\tilde{V}_L^\sigma)_1$ is isomorphic to the semisimple Lie algebra of type $E_6G_2^3$.*

Proof. By Lemma 2.2.2, $\mathfrak{g}^{(234)}$ is an ideal of $(\tilde{V}_L^\sigma)_1$. By the same argument as in Corollary 3.3.4, along with Proposition A.1.1, we have $\dim(\tilde{V}_L^\sigma)_1 = 78 + 8 \times 3 + 9 \times 2 = 120$. Here we set

$$\mathfrak{g} = (\tilde{V}_L^\sigma)_1 / \mathfrak{g}^{(234)}$$

and denote by $\pi : (\tilde{V}_L^\sigma)_1 \rightarrow \mathfrak{g}$ the canonical projection. Then \mathfrak{g} is a semisimple Lie algebra satisfying the following conditions:

- (i) $\dim \mathfrak{g} = 42$;
- (ii) \mathfrak{g} contains a semisimple Lie subalgebra $\pi(\mathfrak{s})$ of type A_2^3 ;
- (iii) the ratio \check{h}/k of every simple component in \mathfrak{g} is equal to $(120 - 24)/24 = 4$, where \check{h} is the dual Coxeter number and k is the level of its affine representation.

Notice that the level k is a positive integer by [DM3, Theorem 1.1]. Since \check{h} is a positive integer, it is a multiple of 4 by (iii). Hence, by (i), \mathfrak{g} is isomorphic to the semisimple Lie algebra of type G_2^3 or C_3^2 . In particular, the rank of \mathfrak{g} is equal to 6. By the conjugacy of Cartan subalgebras and (ii), the root system of \mathfrak{g} contains A_2^3 , which implies that \mathfrak{g} is of type G_2^3 . Thus we have proved the proposition. \square

References

- [ATLAS] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, Atlas of finite groups, Oxford, Oxford University Press 1985.
- [CS] J.H. Conway and N.J.A. Sloane, Sphere packing, lattices and groups, Third edition, Grundlehren der Mathematischen Wissenschaften, Vol. 290, Springer-Verlag, New York, 1999.
- [DGM] L. Dolan, P. Goddard and P. Montague, Conformal field theories, representations and lattice constructions, *Comm. Math. Phys.* **179** (1996), 61–120.
- [D] C. Dong, Vertex algebras associated with even lattice, *J. Algebra* **160** (1993), 245–265.
- [DL1] C. Dong and J. Lepowsky, Generalized vertex algebras and relative vertex operators, Progress in Mathematics Vol. 112, Birkhäuser Boston, Inc., Boston, MA, 1993.
- [DL2] C. Dong and J. Lepowsky, The algebraic structure of relative twisted vertex operators, *J. Pure Appl. Algebra* **110** (1996), 259–295.
- [DLM1] C. Dong, H. Li, and G. Mason, Twisted representations of vertex operator algebras, *Math. Ann.* **310** (1998), 571–600.

- [DLM2] C. Dong, H. Li, and G. Mason, Modular-invariance of trace functions in orbifold theory and generalized Moonshine, *Comm. Math. Phys.* **214** (2000), 1–56.
- [DM1] C. Dong and G. Mason, Holomorphic vertex operator algebras of small central charge, *Pacific J. Math.* **213** (2004), 253–266.
- [DM2] C. Dong and G. Mason, Rational vertex operator algebras and the effective central charge, *Int. Math. Res. Not.* (2004), 2989–3008.
- [DM3] C. Dong and G. Mason, Integrability of C_2 -cofinite vertex operator algebras. *Int. Math. Res. Not.* (2006), Art. ID 80468, 15 pp.
- [DN] C. Dong and K. Nagatomo, Automorphism groups and twisted modules for lattice vertex operator algebras, in “Recent developments in quantum affine algebras and related topics”, Contemp. Math. Vol. 248, pp.117–133, Amer. Math. Soc., Providence, RI, 1999.
- [FLM] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator algebras and the Monster, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.
- [H] J.E. Humphreys, Introduction to Lie algebras and representation theory, Graduate Texts in Mathematics Vol. 9, Springer-Verlag, New York–Berlin, 1978.
- [K] V.G. Kac, Infinite-dimensional Lie algebras, Third edition, Cambridge University Press, Cambridge, 1990
- [La] C.H. Lam, On the constructions of holomorphic vertex operator algebras of central charge 24, *Comm. Math. Phys.* **305** (2011), 153–198.
- [LS1] C.H. Lam and H. Shimakura, Quadratic spaces and holomorphic framed vertex operator algebras of central charge 24, *Proc. Lond. Math. Soc. (3)* **104** (2012), 540–576.
- [LS2] C.H. Lam and H. Shimakura, Classification of holomorphic framed vertex operator algebras of central charge 24, preprint, arXiv:1209.4677.
- [LY] C.H. Lam and H. Yamauchi, On the structure of framed vertex operator algebras and their pointwise frame stabilizers, *Comm. Math. Phys.* **277** (2008), 237–285.
- [Le] J. Lepowsky, Calculus of twisted vertex operators, *Proc. Nat. Acad. Sci. U.S.A.* **82** (1985), 8295–8299.
- [LL] J. Lepowsky and H. Li, Introduction to vertex operator algebras and their representations, Progress in Mathematics Vol. 227, Birkhäuser Boston, Inc., Boston, MA, 2004.
- [Ma] J. Martinet, Perfect lattices in Euclidean spaces, Grundlehren der Mathematischen Wissenschaften Vol. 327, Springer-Verlag, Berlin, 2003.
- [Mi] M. Miyamoto, A \mathbb{Z}_3 -orbifold theory of lattice vertex operator algebra and \mathbb{Z}_3 -orbifold constructions, in “Symmetries, Integrable Systems and Representations”, Springer Proceedings in Mathematics and Statistics Vol. 40, pp.319–344, Springer-Verlag, London, 2013.
- [S] A.N. Schellekens, Meromorphic $c = 24$ conformal field theories, *Comm. Math. Phys.* **153** (1993), 159–185.
- [Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–302.